# Rational points on varieties, part II (surfaces)

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## 1. Del Pezzo surfaces of degree at least five

- Lang-Nishimura [22, Lemma 1.1].
- Severi-Brauer varieties over a global field satisfy the Hasse principle [22, Theorem 2.4].
- Let $X$ be a projective variety over a field $k$ with a separable closure $k^{\mathrm{s}}$ and absolute Galois group $G_k$. Assume that $X$ has a $k$-rational point. Then the natural injection $\operatorname{Pic}_k X \hookrightarrow (\operatorname{Pic}_{k^{\mathrm{s}}} X_{k^{\mathrm{s}}})^{G_k}$ is an isomorphism.

  **Proof (sketch).** It suffices to show every very ample class $c \in (\operatorname{Pic}_{k^{\mathrm{s}}} X_{k^{\mathrm{s}}})^{G_k}$ is in the image. Let $L = \{D \in c : D \geq 0\}$ be the associated complete linear system. Then $L$ has the structure of a variety over $k$, and over $k^{\mathrm{s}}$ the base change $L_{k^{\mathrm{s}}}$ is isomorphic to $\mathbb{P}^n_{k^{\mathrm{s}}}$. Because $c$ is very ample, $L$ induces a morphism $X \to L^*$, where $L^*$ is the dual of $L$, sending $P$ to the hyperplane $\{D \in L : P \in \operatorname{Supp} D\}$. Since $X$ contains a rational point, we find that $L^*$ has a rational point, corresponding to a divisor $A \in \operatorname{Div}_k L$ whose base change $A_{k^{\mathrm{s}}}$ is linearly equivalent with a hyperplane of $\mathbb{P}^n_{k^{\mathrm{s}}}$. Therefore, the linear system $|A|$ (on $L$!) determines an isomorphism $L \to \mathbb{P}^n$ over $k$, so $L$ has a rational point, corresponding to a divisor $D \in c$ defined over $k$. Alternatively, the fact that $L^*$ has a rational point implies that $L^*$ itself is isomorphic to $\mathbb{P}^n$, hence so is its dual $L$.

  If $k$ is a global field and $X$ has a $k_v$-point for all places $v$ of $k$, then the same conclusion holds.

  **Proof (sketch).** This follows pretty much in the same way, but uses that $L$ is a Severi-Brauer variety, and therefore satisfies the Hasse principle.
- Let $X$ over $k$ be a del Pezzo surface of degree at least 5. If $X$ has a $k$-point, then $X$ is birational to $\mathbb{P}^2$ over $k$. If the degree of $X$ is 5 or 7, then $X$ automatically has a $k$-point. Moreover, if $k$ is global, then $X$ satisfies the Hasse principle. [22, Theorem 2.1].

## 2. A remark on Exercise 3(c) from last week

Because I've never seen this worked out in the literature (they always say "a trivial computation shows"), I thought I'd actually work out how to find all 240 curves in a way that does not cause much of a headache. There are probably even more short-cuts.

Let $\pi\colon X \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ in $r \leq 8$ points $P_1, \ldots, P_r$ in general position. Let $K_X$ be a canonical divisor. Then $\operatorname{Pic} X$ is isomorphic to $\mathbb{Z}^{r+1}$ with basis $L = \pi^*\ell$, $E_1, \ldots, E_r$, where $\ell$ is a line in $\mathbb{P}^2$, and $E_i$ is the exceptional curve above $P_i$. Suppose $C$ is an exceptional curve on $X$. Then $C$ is irreducible and there exist integers $b, a_1, \ldots, a_r$ such that $C$ is linearly equivalent to $bL - \sum_i a_i E_i$. The intersection numbers $C^2 = K_X \cdot C = -1$ yield

$$\sum_i a_i^2 = b^2 + 1 \qquad \text{and} \qquad \sum_i a_i = 3b - 1. \tag{1}$$

Assume that $C$ is not one of the $E_i$. Then $C$ intersects all $E_i$ non-negatively, which implies $a_i = C \cdot E_i \geq 0$. This in turn implies $b \geq 1$. If you assume that $C$ is not the strict transform of a line through two of the eight points, and not of a quadric through five of them either, then these curves also have non-negative intersection with $C$, which yields more interesting inequalities that we will not need.

For each $j$, we will bound $a_j$ in terms of $r$ and $b$. We have

$$\sum_{i \neq j} a_i^2 = b^2 + 1 - a_j^2 \qquad \text{and} \qquad \sum_{i \neq j} a_i = 3b - 1 - a_j.$$

The inequality

$$\frac{\sum_{i \neq j} a_i}{r-1} \leq \sqrt{\frac{\sum_{i \neq j}^r a_i^2}{r-1}} \tag{2}$$

between the arithmetic and quadratic mean therefore implies

$$(3b-1-a_j)^2 \le (r-1)(b^2+1-a_j^2),$$

or, equivalently, (complete the square, viewing as polynomial in $a_j$)

$$(ra_j-(3b-1))^2 \le (r-1)(r(b^2+1)-(3b-1)^2). \tag{3}$$

As any integral solution to (1) for $r \le 8$ extends to a solution for $r=8$ by adding zeros, we may assume $r=8$. The right-hand side of (3) has to be non-negative, which is equivalent to $(b+1)(b-7) \le 0$, so we get $b \le 7$. We consider all cases $b \in \{1,2,\ldots,7\}$ separately. The following table gives, for each $b$, the interval that $a_j$ is contained in for each $j$, by (3), in the second column. Here we used that the right-hand side of (3) may be rounded down to the nearest integral square. The third column lists the integral values in that interval.

| $b$ | $a_j \in$ | $a_j \in$ | $t$ | $8t+9-3b$ |
|---|---|---|---|---|
| 1 | $[-\frac{7}{8}, \frac{11}{8}]$ | $\{0,1\}$ | 0 | 6 |
| 2 | $[-\frac{5}{8}, \frac{15}{8}]$ | $\{0,1\}$ | 0 | 3 |
| 3 | $[-\frac{2}{8}, \frac{18}{8}]$ | $\{0,1,2\}$ | | |
| 4 | $[\frac{1}{8}, \frac{21}{8}]$ | $\{1,2\}$ | 1 | 5 |
| 5 | $[\frac{5}{8}, \frac{23}{8}]$ | $\{1,2\}$ | 1 | 2 |
| 6 | $[\frac{10}{8}, \frac{24}{8}]$ | $\{2,3\}$ | 2 | 7 |
| 7 | $\{\frac{5}{2}\}$ | $\emptyset$ | | |

Given that $a_j$ is integral, we find that for $b=7$ there are no solutions. For $b \notin \{3,7\}$, there are only two possible values for $a_j$, say $t$ and $t+1$. If we let $n$ denote the number of $j$ with $a_j = t$, then we obtain

$$3b-1 = \sum_i a_i = nt + (r-n)(t+1) = rt + r - n,$$

so $n = rt + r + 1 - 3b = 8t + 9 - 3b$, which is listed in the table as well. Indeed, all these solutions also satisfy $\sum_i a_i^2 = b^2+1$.

For $b=3$, we can apply a trick that actually works for any $b \le 5$. We have $0 \le a_j \le 2$, so $a_j^2 - a_j$ is nonzero if and only if $a_j = 2$, in which case we have $a_j^2 - a_j = 2$. Hence, the identity

$$\sum_i (a_i^2 - a_i) = (b^2+1) - (3b-1) = b^2 - 3b + 2$$

shows that for exactly $\frac{1}{2}(b^2-3b+2)$ of the indices $j$ we have $a_j = 2$. The number of $j$ with $a_j = 1$ then equals $(\sum_i a_i) - \frac{1}{2}(b^2-3b+2)\cdot 2 = -b^2+6b-3$. Again these all do indeed give solutions to (1).

This yields the following table, containing the types of divisor classes $[C]$ with $C^2 = C \cdot K_X = -1$, and the number of such classes for each $r$.

| $(b; a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$ | $r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $(0; -1, 0, 0, 0, 0, 0, 0, 0)$ | $r$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| $(1; 1, 1, 0, 0, 0, 0, 0, 0)$ | $\binom{r}{2}$ | 0 | 1 | 3 | 6 | 10 | 15 | 21 | 28 |
| $(2; 1, 1, 1, 1, 1, 0, 0, 0)$ | $\binom{r}{5}$ | 0 | 0 | 0 | 0 | 1 | 6 | 21 | 56 |
| $(3; 2, 1, 1, 1, 1, 1, 1, 0)$ | | 0 | 0 | 0 | 0 | 0 | 0 | 7 | 56 |
| $(4; 2, 2, 2, 1, 1, 1, 1, 1)$ | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 56 |
| $(5; 2, 2, 2, 2, 2, 2, 1, 1)$ | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 28 |
| $(6; 3, 2, 2, 2, 2, 2, 2, 2)$ | | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 8 |
| total | | 1 | 3 | 6 | 10 | 16 | 27 | 56 | 240 |

To see that these curves indeed correspond to 240 actual exceptional curves, we first use Riemann-Roch to show that the classes contain an effective divisor. For each $C$, we have

$$\ell(C) - s(C) + \ell(K_X - C) = \tfrac{1}{2}C(C-K_X) + 1 + p_a(X) = \tfrac{1}{2}(C^2 - C\cdot K_X) + 1 = \tfrac{1}{2}(-1+1)+1 = 1,$$

so $\ell(C) + \ell(K_X - C) \ge 1$, so $\ell(C) \ge 1$ or $\ell(K_X - C) \ge 1$, which implies that $C$ is linearly equivalent to an effective curve, or $K_X - C$ is. However, the ample divisor $-K_X$ intersects every effective divisor positively, so the inequality $-K_X \cdot (K_X - C) = -K_X^2 - 1 < 0$ shows that $K_X - C$ is not linearly equivalent to an effective divisor. We conclude that $C$ is, so each of the divisor classes that we found does indeed contain an effective divisor. Because the ample divisor $-K_X$ intersects each component of such a divisor positively, and we have $(-K_X) \cdot C = 1$, we also find that these divisors are prime/irreducible. Their arithmetic genus satisfies

$$2p_a(C) - 2 = C \cdot (C + K_X) = C^2 + C \cdot K_X = -2,$$

so $p_a(C) = 0$, which implies that $C$ is smooth.

Finally, each class contains a unique effective curve, as two different irreducible curves can not intersect negatively.

Hence, we really have $1, 3, 6, 10, 16, 27, 56, 240$ exceptional curves for $r = 1, 2, 3, 4, 5, 6, 7, 8$, respectively.

## 3. Del Pezzo surfaces of degree 6

We now sketch an alternative completion of the proof of the fact that Del Pezzo surfaces of degree 6 over a global field satisfy the Hasse principle.

Let $X$ be a variety over a field $k$ and $m$ a positive integer. Then there exists a variety $\mathrm{Sym}^m X$ over $k$ of which the $\overline{k}$-points are the orbits of $\prod_{i=1}^m X(\overline{k})$ under the action of the permutation group $S_m$, acting by permuting the $m$ factors. Moreover, if $X$ is smooth, then $\mathrm{Sym}^m X$ is smooth at all points corresponding to orbits of $m$-tuples of $m$ different points. You may use this in the exercises below as well.

Let $X$ be a del Pezzo surface of degree 6, embedded anticanonically in $\mathbb{P}^6$. Let

$$z = ([(Q, Q', Q'')], [(R, R')]) \in (\mathrm{Sym}^3 X) \times (\mathrm{Sym}^2 X)$$

be a point for which $Q, Q', Q'', R, R'$ are five different points. Let $M_z$ be the 4-dimensional linear subspace of $\mathbb{P}^6$ spanned by these five points. If $z$ is general enough, then the intersection $M_z \cap X$ is 0-dimensional; it then has degree 6, with five intersection points already known, so the sixth intersection point is unique. This yields a rational map

$$(\mathrm{Sym}^3 X) \times (\mathrm{Sym}^2 X) \dashrightarrow X,$$

sending $z$ to the sixth intersection points of $M_z \cap X$.

Now let $K$ and $L$ be separable field extensions of $k$ of degrees 2 and 3, respectively. Suppose $X(K)$ and $X(L)$ are not empty, say $Q \in X(L)$ and $R \in X(K)$. If $Q$ or $R$ is defined over $k$, then $X(k)$ is not empty. Otherwise, let $Q'$ and $Q''$ be the conjugates of $Q$ and $R'$ the conjugate of $R$. Then $z = ([(Q, Q', Q'')], [(R, R')]) \in (\mathrm{Sym}^3 X) \times (\mathrm{Sym}^2 X)$ is a smooth point over $k$. Since $X$ is proper, we find by Lang-Nishimura that $X$ also has a $k$-rational point.

Together with what we did in class, this proves that $X$ satisfies the Hasse principle if $k$ is a global field. For the proof in class, see [22, 2.4. case 4.] and the references given there. For this alternative proof, see [17, 9.4.4].

## 4. Exercises

(1) Suppose $X$ is a del Pezzo surface of degree 5 over a field $k$. Let $P \in X(k)$ be a point that lies on (at least) one of the 10 exceptional curves of $X$. Show that $X$ is not minimal, i.e., there exists a Galois stable set of exceptional curves that pairwise do not intersect (which can be blown down over $k$, hence the terminology "not minimal").

(2) Let $X$ be a del Pezzo surface of degree $d \geq 3$. Suppose that $X$ has a point over a separable field extension $K$ of $k$ of degree $[K : k] = d - 1$. Show that $X$ also has a $k$-rational point.

(3) Email me before Monday, December 9, with the times on Monday, December 16, that you can **not** do the oral exam.

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