(1) Which of the following are linear subspaces of the vector space $\mathbb{R}^2$? Explain your answers!
  (a) $U_1 = \{(x, y) \in \mathbb{R}^2 : y = -\sqrt{e^\pi} x\}$
  (b) $U_2 = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$
  (c) $U_3 = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$

(2) Which of the following are linear subspaces of the vector space V of all functions from $\mathbb{R}$ to $\mathbb{R}$?
  (a) $U_1 = \{f \in V \ : \ f \text{ is continuous}\}$
  (b) $U_2 = \{f \in V \ : \ f(3) = 0\}$
  (c) $U_3 = \{f \in V \ : \ f \text{ is continuous or } f(3) = 0\}$
  (d) $U_4 = \{f \in V \ : \ f \text{ is continuous and } f(3) = 0\}$
  (e) $U_5 = \{f \in V \ : \ f(0) = 3\}$
  (f) $U_6 = \{f \in V \ : \ f(0) \geq 0\}$

(3) Given a vector space V with subsets I and J of V, show that

$$L(I \cup J) = L(I) + L(J).$$

Does the equality $L(I \cap J) = L(I) \cap L(J)$ hold?

(4) Which of the following sequences of vectors in $\mathbb{R}^3$ are linear independent?
  (a) $\big((1, 2, 3), (2, 1, -1), (-1, 1, 1)\big)$
  (b) $\big((1, 3, 2), (1, 1, 1), (-1, 3, 1)\big)$

(5) Let F be a field, $n > 0$ an integer, and set $V = F^n$. For $i \in \{1, \ldots, n\}$, let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the vector with all zeroes, except for a 1 at the i-th position. Show that $(e_1, e_2, \ldots, e_n)$ is a basis for F^n.

(6) For any positive integer $n > 0$, let P_n be the vector space of polynomials in x over the field F of degree at most n; show that $(1, x, x^2, \ldots, x^n)$ a basis is for P_n. Show that $(1, x - 1, (x - 1)^2, \ldots, (x - 1)^n)$ also basis is (Hint: consider the degree).

(7) Give a basis for each of the following $\mathbb{R}$-vector spaces.

$$U_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - x_2 + x_3 = 0\}$$
$$U_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + 2x_2 + 3x_3 = 0,\ x_1 + x_2 - x_4 = 0\}$$

Use the definition of 'basis' from the lecture to show your answer is correct.

(8) Let U_1 and U_2 be two linear subspaces of a vector space V.
Show that $U_1 \cup U_2$ is a linear subspace of V if and only if $U_1 \subset U_2$ or $U_2 \subset U_1$.

(9) Give examples of a vector space V and linear subspaces $U_1, U_2, U_3 \subset V$ that show that in general
  (a) $(U_1 \cap U_2) + U_3 \neq (U_1 + U_3) \cap (U_2 + U_3)$
  (b) $(U_1 + U_2) \cap U_3 \neq (U_1 \cap U_3) + (U_2 \cap U_3)$

(10) Let V be a vector space with $\dim V = n$, and take $v_1, \ldots, v_n \in V$. Prove that the following statements are equivalent.
  (i) $\{v_1, \ldots, v_n\}$ is a basis of V
  (ii) $v_1, \ldots, v_n$ are linearly independent
  (iii) $L(v_1, \ldots, v_n) = V$

(11) Let V be a vector space and $v_1, \ldots, v_n \in V$. Show that $\dim L(v_1, \ldots, v_n) \leq n$.

(12) Let V be a real vector space, and $a, b, c, d \in V$. Show that the following vectors are linearly dependent:

$$\begin{aligned} v_1 &= 2a && && &&+ 9d \\ v_2 &= && && 5c && \\ v_3 &= a + && b + && c + && d \\ v_4 &= a + && 2b + && 3c + && 4d \\ v_5 &= a && && && \end{aligned}$$

HINT. There is a very short solution.

(13) Give a basis for each of the following $\mathbb{R}$-vector spaces.

$U_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - x_2 + x_3 = 0\}$

$U_2 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + 2x_2 + 3x_3 = 0,\ x_1 + x_2 - x_4 = 0\}$

Use the definition of 'basis' from the lecture to show your answer is correct.

(14) Let $P_n(F)$ denote the vector space of polynomials of degree at most n. For each $k \in \mathbb{Z}_{\geq 0}$, define

$$\binom{x}{k} = \frac{1}{k!} x(x-1)(x-2)\cdots(x-k+1).$$

Show that

$$\left(\binom{x}{0}, \binom{x}{1}, \binom{x}{2}, \ldots \binom{x}{n-1}, \binom{x}{n}\right)$$

is a basis for $P_n(F)$.

(15) Show that the vectors

$$v_1 = (1, 2, 3, 4), \quad v_2 = (1, 1, 1, 1), \text{ and } v_3 = (1, 1, 0, -1)$$

in $\mathbb{Q}^4$ are linearly independent and extend (v_1, v_2, v_3) to a basis for $\mathbb{Q}^4$.