Linear Algebra I

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1. Vector spaces

1.1. **Examples.**

1.2. **Fields.**

Exercise 1.2.1. Prove Proposition **??**.

Exercise 1.2.2. Check that $\mathbb{F}_2$ is a field (see Example **??**).

Exercise 1.2.3. Which of the following are fields?

(1) The set $\mathbb{N}$ together with the usual addition and multiplication.
(2) The set $\mathbb{Z}$ together with the usual addition and multiplication.
(3) The set $\mathbb{Q}$ together with the usual addition and multiplication.
(4) The set $\mathbb{R}_{\geq 0}$ together with the usual addition and multiplication.
(5) The set $\mathbb{Q}(\sqrt{3}) = \{a + b\sqrt{3} \ : \ a, b \in \mathbb{Q}\}$ together with the usual addition and multiplication.
(6) The set $\mathbb{F}_3 = \{0, 1, 2\}$ with the usual addition and multiplication, followed by taking the remainder after division by 3.

1.3. **The field of complex numbers.**

Exercise 1.3.1. Prove Remark **??**.

Exercise 1.3.2. For every complex number z we have $\mathrm{Re}(z) = \frac{1}{2}(z + \overline{z})$ and $\mathrm{Im}(z) = \frac{1}{2i}(z - \overline{z})$.

1.4. **Definition of a vector space.**

Exercise 1.4.1. Compute the inner product of the given vectors v and w in $\mathbb{R}^2$ and draw a corresponding picture (cf. Example **??**).

(1) $v = (-2, 5)$ and $w = (7, 1)$,
(2) $v = 2(-3, 2)$ and $w = (1, 3) + (-2, 4)$,
(3) $v = (-3, 4)$ and $w = (4, 3)$,
(4) $v = (-3, 4)$ and $w = (8, 6)$,
(5) $v = (2, -7)$ and $w = (x, y)$,
(6) $v = w = (a, b)$.

Exercise 1.4.2. Write the following equations for lines in $\mathbb{R}^2$ with coordinates x_1 and x_2 in the form $\langle a, x\rangle = c$, i.e., specify a vector a and a constant c in each case.

(1) $L_1 \colon 2x_1 + 3x_2 = 0$,
(2) $L_2 \colon x_2 = 3x_1 - 1$,
(3) $L_3 \colon 2(x_1 + x_2) = 3$,
(4) $L_4 \colon x_1 - x_2 = 2x_2 - 3$,
(5) $L_5 \colon x_1 = 4 - 3x_1$,
(6) $L_6 \colon x_1 - x_2 = x_1 + x_2$.
(7) $L_7 \colon 6x_1 - 2x_2 = 7$

Exercise 1.4.3. True or False? If true, explain why. If false, give a counterexample.

(1) If $a, b \in \mathbb{R}^2$ are nonzero vectors and $a \neq b$, then the lines in $\mathbb{R}^2$ given by $\langle a, x\rangle = 0$ and $\langle b, x\rangle = 1$ are not parallel.
(2) If $a, b \in \mathbb{R}^2$ are nonzero vectors and the lines in $\mathbb{R}^2$ given by $\langle a, x\rangle = 0$ and $\langle b, x\rangle = 1$ are parallel, then $a = b$.

(3) Two different hyperplanes in F^n may be given by the same equation.
(4) The intersection of two lines in F^n is either empty or consists of one point.
(5) For each vector $v \in \mathbb{R}^2$ we have $0 \cdot v = 0$. (What do the zeros in this statement refer to?)

Exercise 1.4.4. In Example **??**, the first distributive law and the existence of negatives were proved for F^n. Show that the other six axioms for vector spaces hold for F^n as well, so that F^n is indeed a vector space over F.

Exercise 1.4.5. In Example **??**, the first distributive law was proved for F^X. Show that the other seven axioms for vector spaces hold for F^X as well, so that F^X is indeed a vector space over F.

Exercise 1.4.6. Let $(V, 0, +, \cdot)$ be a real vector space and define $x - y = x + (-y)$, as usual. Which of the vector space axioms are satisfied and which are not (in general), for $(V, 0, -, \cdot)$?

NOTE. You are expected to give proofs for the axioms that hold and to give counterexamples for those that do not hold.

Exercise 1.4.7. Prove that the set $P(F)$ of polynomials over F, together with addition, scalar multiplication, and the zero as defined in Example **??** is a vector space.

Exercise 1.4.8. Given the field F and the set V in the following cases, together with the described addition and scalar multiplication, as well as the implicit element 0, which cases determine a vector space? If not, then which rule is not satisfied?

(1) The field $F = \mathbb{R}$ and the set V of all functions $[0,1] \to \mathbb{R}_{>0}$, together with the usual addition and scalar multiplication.
(2) Example **??**.
(3) The field $F = \mathbb{Q}$ and the set $V = \mathbb{R}$ with the usual addition and multiplication.
(4) The field $\mathbb{R}$ and the set V of all functions $f \colon \mathbb{R} \to \mathbb{R}$ with $f(3) = 0$, together with the usual addition and scalar multiplication.
(5) The field $\mathbb{R}$ and the set V of all functions $f \colon \mathbb{R} \to \mathbb{R}$ with $f(3) = 1$, together with the usual addition and scalar multiplication.
(6) Any field F together with the subset

$$\{(x, y, z) \in F^3 \ : \ x + 2y - z = 0\},$$

with coordinatewise addition and scalar multiplication.
(7) The field $F = \mathbb{R}$ together with the subset

$$\{(x, y, z) \in F^3 \ : \ x - z = 1\},$$

with coordinatewise addition and scalar multiplication.

Exercise 1.4.9. Suppose the set X contains exactly n elements. Then how many elements does the vector space $\mathbb{F}_2^X$ of functions $X \to \mathbb{F}_2$ consist of?

Exercise 1.4.10. We can generalize Example **??** further. Let F be a field and V a vector space over F. Let X be any set and let $V^X = \mathrm{Map}(X, V)$ be the set of all functions $f \colon X \to V$. Define an addition and scalar multiplication on V^X that makes it into a vector space.

Exercise 1.4.11. Let S be the set of all sequences $(a_n)_{n\geq 0}$ of real numbers satisfying the recurrence relation

$$a_{n+2} = a_{n+1} + a_n \quad \text{for all } n \geq 0.$$

Show that the (term-wise) sum of two sequences from S is again in S and that any (term-wise) scalar multiple of a sequence from S is again in S. Finally show that S (with this addition and scalar multiplication) is a real vector space.

Exercise 1.4.12. Let U and V be vector spaces over the same field F. Consider the Cartesian product

$$W = U \times V = \{\, (u, v) \ : \ u \in U, \, v \in V \,\}.$$

Define an addition and scalar multiplication on W that makes it into a vector space.

***Exercise 1.4.13.** For each of the eight axioms in Definition **??**, try to find a system $(V, 0, +, \cdot)$ that does not satisfy that axiom, while it does satisfy the other seven.

1.5. **Basic properties.**

Exercise 1.5.1. Proof Proposition **??**.

Exercise 1.5.2. Proof Remarks **??**.

Exercise 1.5.3. Is the following statement correct? “Axiom (4) of Definition **??** is redundant because we already know by Remarks ??(2) that for each vector $x \in V$ the vector $-x = (-1) \cdot x$ is also contained in V.”

2. Subspaces

2.1. **Definition and examples.**

Exercise 2.1.1. Given an integer $d \geq 0$, let $P_d(\mathbb{R})$ denote the set of polynomials of degree at most d. Show that the addition of two polynomials $f, g \in P_d(\mathbb{R})$ satisfies $f + g \in P_d(\mathbb{R})$. Show also that any scalar multiple of a polynomial $f \in P_d(\mathbb{R})$ is contained in $P_d(\mathbb{R})$. Prove that $P_d(\mathbb{R})$ is a vector space.

Exercise 2.1.2. Let X be a set with elements $x_1, x_2 \in X$, and let F be a field. Is the set

$$U = \{\, f \in F^X \ : \ f(x_1) = 2f(x_2) \,\}$$

a subspace of F^X?

Exercise 2.1.3. Let X be a set with elements $x_1, x_2 \in X$. Is the set

$$U = \{\, f \in \mathbb{R}^X \ : \ f(x_1) = f(x_2)^2 \,\}$$

a subspace of $\mathbb{R}^X$?

Exercise 2.1.4. Which of the following are linear subspaces of the vector space $\mathbb{R}^2$? Explain your answers!

(1) $U_1 = \{(x, y) \in \mathbb{R}^2 : y = -\sqrt{e^\pi} x\}$,
(2) $U_2 = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$,
(3) $U_3 = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$.

Exercise 2.1.5. Which of the following are linear subspaces of the vector space V of all functions from $\mathbb{R}$ to $\mathbb{R}$?

(1) $U_1 = \{f \in V \ : \ f \text{ is continuous}\}$
(2) $U_2 = \{f \in V \ : \ f(3) = 0\}$
(3) $U_3 = \{f \in V \ : \ f \text{ is continuous or } f(3) = 0\}$
(4) $U_4 = \{f \in V \ : \ f \text{ is continuous and } f(3) = 0\}$
(5) $U_5 = \{f \in V \ : \ f(0) = 3\}$
(6) $U_6 = \{f \in V \ : \ f(0) \geq 0\}$

Exercise 2.1.6. Prove Proposition **??**.

Exercise 2.1.7. Prove Proposition **??**.

Exercise 2.1.8. Let F be any field. Let $a_1, \ldots, a_t \in F^n$ be vectors and $b_1, \ldots, b_t \in F$ constants. Let $V \subset F^n$ be the subset

$$V = \{x \in F^n \ : \ \langle a_1, x\rangle = b_1, \ldots, \langle a_t, x\rangle = b_t\}.$$

Show that with the same addition and scalar multiplication as F^n, the set V is a vector space if and only if $b_1 = \ldots = b_t = 0$.

Exercise 2.1.9.

(1) Let X be a set and F a field. Show that the set $F^{(X)}$ of all functions $f\colon X \to F$ that satisfy $f(x) = 0$ for all but finitely many $x \in X$ is a subspace of the vector space F^X.
(2) More generally, let X be a set, F a field, and V a vector space over F. Show that the set $V^{(X)}$ of all functions $f\colon X \to V$ that satisfy $f(x) = 0$ for all but finitely many $x \in X$ is a subspace of the vector space V^X (cf. Exercise 1.4.10).

2.2. Intersections.

Exercise 2.2.1. Suppose that U_1 and U_2 are linear subspaces of a vector space V. Show that $U_1 \cup U_2$ is a subspace of V if and only if $U_1 \subset U_2$ or $U_2 \subset U_1$.

Exercise 2.2.2. Let H_1, H_2, H_3 be hyperplanes in $\mathbb{R}^3$ given by the equations

$$\langle (1,0,1), v\rangle = 2, \qquad \langle (-1,2,1), v\rangle = 0, \qquad \langle (1,1,1), v\rangle = 3,$$

respectively.

(1) Which of these hyperplanes is a subspace of $\mathbb{R}^3$?
(2) Show that the intersection $H_1 \cap H_2 \cap H_3$ contains exactly one element.

Exercise 2.2.3. Give an example of a vector space V with two subsets U_1 and U_2, such that U_1 and U_2 are **not** subspaces of V, but their intersection $U_1 \cap U_2$ is.

2.3. Linear hulls, linear combinations, and generators.

Exercise 2.3.1. Prove Proposition **??**.

Exercise 2.3.2. Do the vectors

$$(1,0,-1), \qquad (2,1,1), \qquad \text{and} \qquad (1,0,1)$$

generate $\mathbb{R}^3$?

Exercise 2.3.3. Do the vectors

$$(1,2,3), \qquad (4,5,6), \qquad \text{and} \qquad (7,8,9)$$

generate $\mathbb{R}^3$?

Exercise 2.3.4. Let $U \subset \mathbb{R}^4$ be the subspaces generated by the vectors

$$(1,2,3,4), \qquad (5,6,7,8), \qquad \text{and} \qquad (9,10,11,12).$$

What is the minimum number of vectors needed to generate U? As always, prove that your answer is correct.

Exercise 2.3.5. Let F be a field and X a set. Consider the subspace $F^{(X)}$ of F^X consisting of all functions $f\colon X \to F$ that satisfy $f(x) = 0$ for all but finitely many $x \in X$ (cf. Exercise 2.1.9). For every $x \in X$ we define the function $e_x\colon X \to F$ by

$$e_x(z) = \begin{cases} 1 & \text{if } z = x, \\ 0 & \text{otherwise.} \end{cases}$$

Show that the set $\{e_x \ : \ x \in X\}$ generates $F^{(X)}$.

Exercise 2.3.6. Does the equality $L(I \cap J) = L(I) \cap L(J)$ hold for all vector spaces V with subsets I and J of V?

Exercise 2.3.7. We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *even* if $f(-x) = f(x)$ for all $x \in \mathbb{R}$, and *odd* if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

(1) Is the subset of $\mathbb{R}^{\mathbb{R}}$ consisting of al even functions a linear subspace?
(2) Is the subset of $\mathbb{R}^{\mathbb{R}}$ consisting of al odd functions a linear subspace?

Exercise 2.3.8. Given a vector space V over a field F and vectors $v_1, v_2, \ldots, v_n \in V$. Set $W = L(v_1, v_2, \ldots, v_n)$. Using Remark **??**, give short proofs of the following equalities of subspaces.

(1) $W = L(v_1', \ldots, v_n')$ where for some fixed j and k we set $v_i' = v_i$ for $i \neq j, k$ and $v_j' = v_k$ and $v_k' = v_j$ (the elements v_j and v_k are switched),
(2) $W = L(v_1', \ldots, v_n')$ where for some fixed j and some nonzero scalar $\lambda \in F$ we have $v_i' = v_i$ for $i \neq j$ and $v_j' = \lambda v_j$ (the j-th vector is scaled by a nonzero factor λ).
(3) $W = L(v_1', \ldots, v_n')$ where for some fixed j, k with $j \neq k$ and some scalar $\lambda \in F$ we have $v_i' = v_i$ for $i \neq k$ and $v_k' = v_k + \lambda v_j$ (a scalar multiple of v_j is added to v_k).

2.4. **Sums of subspaces.**

Exercise 2.4.1. Prove Lemma **??**.

Exercise 2.4.2. Suppose F is a field and $U_1, U_2 \subset F^n$ subspaces. Show that we have

$$(U_1 + U_2)^\perp = U_1^\perp \cap U_2^\perp.$$

Exercise 2.4.3. Suppose V is a vector space with a subspace $U \subset V$. Suppose that $U_1, U_2 \subset V$ subspaces of V that are contained in U. Show that the sum $U_1 + U_2$ is also contained in U.

Exercise 2.4.4. Take $u = (1,0)$ and $u' = (\alpha, 1)$ in $\mathbb{R}^2$, for any $\alpha \in \mathbb{R}$. Show that $U = L(u)$ and $U' = L(u')$ are complementary subspaces.

Exercise 2.4.5. Let U_+ and U_- be the subspaces of $\mathbb{R}^{\mathbb{R}}$ of even and odd functions, respectively (cf. Exercise 2.3.7).

(1) Show that for any $f \in \mathbb{R}^{\mathbb{R}}$, the functions f_+ and f_- given by

$$f_+(x) = \frac{f(x) + f(-x)}{2} \qquad \text{and} \qquad f_-(x) = \frac{f(x) - f(-x)}{2}$$

are even and odd, respectively.

(2) Show that U_+ and U_- are complementary subspaces.

Exercise 2.4.6. Are the subspaces U_0 and U_1 of Example **??** complementary subspaces?

Exercise 2.4.7. True or false? For every subspaces U, V, W of a common vector space, we have $U \cap (V + W) = (U \cap V) + (U \cap W)$. Prove it, or give a counterexample.

2.5. **Euclidean space.**

Exercise 2.5.1. Prove Lemma **??**.

Exercise 2.5.2. Take $a = (-1, 2, 1) \in \mathbb{R}^3$ and set $V = \{a\}^\perp \subset \mathbb{R}^3$. Write the element $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ as $x = x' + x''$ with $x \in L(a)$ and $x'' \in V$.

Exercise 2.5.3. Finish the proof of Proposition **??**.

Exercise 2.5.4. Explain why Proposition **??** might be called the triangle inequality, which usually refers to $c \leq a + b$ for the sides a, b, c of a triangle. Prove that for all $v, w \in \mathbb{R}^n$ we have $\|v - w\| \leq \|v\| + \|w\|$.

Exercise 2.5.5. Prove the cosine rule in $\mathbb{R}^n$.

Exercise 2.5.6. Show that two vectors $v, w \in \mathbb{R}^n$ have the same length if and only $v - w$ and $v + w$ are orthogonal.

Exercise 2.5.7. Prove that the diagonals of a parallelogram are orthogonal to each other if and only if all sides have the same length.

Exercise 2.5.8. Compute the distance from the point $(1, 1, 1, 1) \in \mathbb{R}^4$ to the line $L(a)$ with $a = (1, 2, 3, 4)$.

Exercise 2.5.9. Given the vectors $p = (1, 2, 3)$ and $w = (2, 1, 5)$, let L be the line consisting of all points of the form $p + \lambda w$ for some $\lambda \in \mathbb{R}$. Compute the distance $d(v, L)$ for $v = (2, 1, 3)$.

Exercise 2.5.10. Let $H \subset \mathbb{R}^4$ be the hyperplane with normal $a = (1, -1, 1, -1)$ going though the point $q = (1, 2, -1, -2)$. Determine the distance from the point $(2, 1, -3, 1)$ to H.

Exercise 2.5.11. Determine the angle between the vectors $(1, -1, 2)$ and $(-2, 1, 1)$ in $\mathbb{R}^3$.

Exercise 2.5.12. The angle between two hyperplanes is defined as the angle between their normal vectors. Determine the angle between the hyperplanes in $\mathbb{R}^4$ given by $x_1 - 2x_2 + x_3 - x_4 = 2$ and $3x_1 - x_2 + 2x_3 - 2x_4 = -1$, respectively.

3. Linear Maps

3.1. **Review of maps.**

3.2. **Definition and examples.**

Exercises.

Exercise 3.2.1. Prove Lemma **??**.

Exercise 3.2.2. Welke van de volgende functies tussen vectorruimtes is een lineaire afbeelding?

(1) $\mathbb{R}^3 \to \mathbb{R}^2,\ (x, y, z) \mapsto (x - 2y, z + 1)$,
(2) $\mathbb{R}^3 \to \mathbb{R}^3,\ (x, y, z) \mapsto (x^2, y^2, z^2)$,
(3) $\mathbb{C}^3 \to \mathbb{C}^4,\ (x, y, z) \mapsto (x + 2y, x - 3z, y - z, x + 2y + z)$,
(4) $\mathbb{R}^3 \to V,\ (x, y, z) \mapsto xv_1 + yv_2 + zv_3$, voor een vectorruimte V over $\mathbb{R}$ met $v_1, v_2, v_3 \in V$,
(5) $P(\mathbb{C}) \to P(\mathbb{C}),\ f \mapsto f'$, waarbij $P(\mathbb{C})$ de vectorruimte van polynomen over $\mathbb{C}$ is en f' de afgeleide van f,
(6) $P(\mathbb{R}) \to \mathbb{R}^2,\ f \mapsto (f(2), f'(0))$.

Exercise 3.2.3. Let $f \colon V \to W$ be a linear map of vector spaces. Show that the following are equivalent.

(1) The map f is surjective.
(2) For every subset $S \subset V$ with $L(S) = V$ we have $L(f(S)) = W$.
(3) There is a subset $S \subset V$ with $L(f(S)) = W$.

Exercise 3.2.4. Zij $\rho \colon \mathbb{R}^2 \to \mathbb{R}^2$ de afbeelding gegeven door rotatie om de oorsprong $(0, 0)$ over de hoek θ.

(1) Show that ρ is a linear map.
(2) Wat zijn de beelden $\rho((1, 0))$ en $\rho((0, 1))$?
(3) Laat zien dat er geldt

$$\rho((x, y)) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta).$$

Exercise 3.2.5. Show that the reflection $s \colon \mathbb{R}^2 \to \mathbb{R}^2$ in the line given by $y = -x$ is a linear map. Give an explicit formule for s.

Exercise 3.2.6. Gegeven is de afbeelding

$$T \colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto x(\tfrac{3}{5}, \tfrac{4}{5}) + y(\tfrac{4}{5}, -\tfrac{3}{5})$$

en de vectoren $v_1 = (2, 1)$ en $v_2 = (-1, 2)$.

(1) Laat zien dat er geldt $T(v_1) = v_1$ en $T(v_2) = -v_2$.
(2) Laat zien dat de lineaire afbeelding gegeven door spiegeling in de lijn $2y - x = 0$ gelijk is aan T.

Exercise 3.2.7. Geef een expliciete uitdrukking voor de lineaire afbeelding $s \colon \mathbb{R}^2 \to \mathbb{R}^2$ die gegeven wordt door spiegeling in de lijn $y = 3x$.

Exercise 3.2.8. Let F be a field and n a nonnegative integer. Show that there is an isomorphism

$$F^n \to \mathrm{Hom}(F^n, F)$$

that sends a vector $a \in F^n$ to te linear map $x \mapsto \langle a, x\rangle$.

4. Matrices

4.1. **Definition of matrices.**

4.2. **Linear maps associated to matrices.**

Exercises.

Exercise 4.2.1. Prove Lemma **??** using the column vectors of A.

Exercise 4.2.2. Prove Remark **??**.

Exercise 4.2.3. Voor de gegeven matrix A en vector x, bereken Ax.

(1)
$$A = \begin{pmatrix} -2 & -3 & 1 \\ 1 & 1 & -2 \\ 0 & 1 & 1 \end{pmatrix} \qquad \text{en} \qquad x = \begin{pmatrix} -3 \\ -4 \\ 2 \end{pmatrix},$$

(2)
$$A = \begin{pmatrix} 1 & -3 & 2 \\ -2 & -4 & 2 \end{pmatrix} \qquad \text{en} \qquad x = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix},$$

(3)
$$A = \begin{pmatrix} 4 & 3 \\ 3 & -2 \\ -3 & -1 \\ -1 & 1 \end{pmatrix} \qquad \text{en} \qquad x = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

Exercise 4.2.4. Geef voor elk van de lineaire afbeeldingen $f\colon F^n \to F^m$ van de opgaven van het vorige hoofdstuk een matrix M zodat f gegeven wordt door
$$x \mapsto Mx.$$

Exercise 4.2.5. Gegeven de matrix
$$M = \begin{pmatrix} -4 & -3 & 0 & -3 \\ 2 & 2 & -3 & -1 \\ 0 & -3 & 1 & -1 \end{pmatrix}$$
en de lineaire afbeelding $f\colon \mathbb{R}^n \to \mathbb{R}^m, x \mapsto Mx$ for de bijbehorende m en n. Wat zijn m en n en wat zijn vectoren $v_1, \ldots, v_n$ zodanig dat f ook gegeven wordt door
$$f\big((x_1, x_2, \ldots, x_n)\big) = x_1 v_1 + \ldots + x_n v_n?$$

Exercise 4.2.6. Voor welke $i, j \in \{1, \ldots, 5\}$ bestaat het product $A_i \cdot A_j$ en in welke volgorde?
$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -2 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 2 & -1 & 1 & -4 \\ 3 & -1 & 2 & 4 \end{pmatrix}$$
$$A_3 = \begin{pmatrix} 2 & 3 & 4 \\ -1 & 0 & 2 \\ 3 & 2 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -1 & -3 \\ 2 & -2 \\ 1 & 1 \end{pmatrix}, \quad A_5 = \begin{pmatrix} 1 & -2 \\ -3 & 2 \end{pmatrix}.$$
Bereken (een aantal van) deze producten.

Exercise 4.2.7. Voor elke $i \in \{1, , \ldots, 5\}$ definiëren we de lineaire afbeelding f_i door $x \mapsto A_i x$ met A_i als in de vorige opgave.

(1) Wat zijn de domeinen en codomeinen van deze functies?

(2) Welke van deze functies kun je samenstellen en welke matrices horen dan bij de samenstelling (geef alleen aan welke twee matrices je moet vermenigvuldigen en in welke volgorde)?
(3) Is er een volgorde waarop je alle functies kunt samenstellen, en zo ja, welk product van matrices hoort bij deze samenstelling, en wat is het domein en codomein?

Exercise 4.2.8. Vind twee matrices A en B zodanig dat AB een nulmatrix is (die dus alleen maar nullen bevat), terwijl het product BA ook bestaat, maar geen nulmatrix is.

Exercise 4.2.9. Gegeven de volgende lineaire afbeeldingen $\mathbb{R}^n \to \mathbb{R}^m$, bepaal een matrix A zodanig dat de afbeelding ook geschreven kan worden als $x \mapsto Ax$.

(1) $f\colon \mathbb{R}^3 \to \mathbb{R}^4,\ (x, y, z) \mapsto (3x + 2y - z, -x - y + z, x - z, y + z)$,
(2) $g\colon \mathbb{R}^3 \to \mathbb{R}^3,\ (x, y, z) \mapsto (x + 2y - 3z, 2x - y + z, x + y + z)$,
(3) $h\colon \mathbb{R}^3 \to \mathbb{R}^2,\ (x, y, z) \mapsto x \cdot (1, 2) + y \cdot (2, -1) + z \cdot (-1, 3)$,
(4) $j\colon \mathbb{R}^2 \to \mathbb{R}^3,\ v \mapsto (\langle v, w_1\rangle, \langle v, w_2\rangle, \langle v, w_3\rangle)$, met $w_1 = (1, -1)$, $w_2 = (2, 3)$ en $w_3 = (-2, 4)$.

Exercise 4.2.10. Neem de lineaire afbeeldingen f en g uit de vorige opgave en noem de bijbehorende matrices A en B. In welke volgorde kun je f en g samenstellen? Stel ze met de hand ook daadwerkelijk samen en schrijf die samenstelling op dezelfde manier op als f en g ook gegeven zijn. Bereken het product van de matrices A en B (in de juiste volgorde) en verifieer dat dit product inderdaad overeen komt met de samenstelling van de lineaire afbeeldingen.

Exercise 4.2.11. Let F be a field and m, n nonnegative integers. Show that there exists an isomorphism

$$\mathrm{Mat}_{m,n}(F) \to \mathrm{Hom}(F^n, F^m)$$

that sends A to f_A. (This is in fact almost true by definition, as we defined the addition and scalar product of matrices in terms of the addition and scalar product of the functions that are associated to them.)