STRUCTURE OF NILPOTENT ENDOMORPHISMS

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This is an alternative proof of Theorem 3.3 in Michael Stoll's "Linear Algebra II" (2007).

Lemma 1. *Let V be a finite-dimensional vector space over a field F.*

a) *Let $U_1, U_2 \subset V$ be subspaces of V satisfying $U_1 \cap U_2 = \{0\}$. Then any basis of U_2 can be extended to a basis of a complementary space of U_1 inside V.*
b) *Let $U_1, U_2, U_3 \subset V$ be subspaces of V such that U_3 is a complementary space of $U_1 + U_2$ inside V, and U_2 is a complementary space of U_1 inside $U_1 + U_2$. Then $U_2 + U_3$ is a complementary space of U_1 inside V and the union of any bases for U_2 and U_3 is a basis for $U_2 + U_3$.*

Proof. Exercise. □

Theorem 2. *Let V be a finite-dimensional vector space over a field F and set $n = \dim V$. Let $f\colon V \to V$ be a nilpotent endomorphism. Then V has a basis $(v_1, \ldots, v_n)$ such that for all $i \in \{1, \ldots, n\}$ we have $f(v_i) = v_{i+1}$ or $f(v_i) = 0$.*

Proof. Let m be an integer such that $f^m = 0$. Note that we have a chain of inclusions

$$\{0\} = \ker f^0 \subset \ker f^1 \subset \ker f^2 \subset \cdots \subset \ker f^{m-1} \subset \ker f^m = V.$$

We prove by descending induction that for all $j \in \{0, 1, \ldots, m\}$ there are elements $w_1, \ldots, w_s \in V$ and non-negative integers $d_1, \ldots, d_s$, such that the sequence

$$\big(w_1, f(w_1), \ldots, f^{d_1}(w_1), w_2, f(w_2), \ldots, f^{d_2}(w_2), \ldots, w_s, f(w_s), \ldots, f^{d_s}(w_s)\big) \tag{1}$$

is a basis of a complementary space X_j of $\ker f^j$ inside V and, if $j > 0$, the sequence

$$\big(f^{d_1+1}(w_1), \ldots, f^{d_s+1}(w_s)\big) \tag{2}$$

is a basis of a subspace Y'_j of $\ker f^j$ satisfying $Y'_j \cap \ker f^{j-1} = \{0\}$.

For $j = m$ this is true because we can take $\ell = 0$ and $X_j = Y'_j = 0$ (the zero space is a complementary space of V inside V). Suppose $0 \leq j < m$ and suppose we have elements $w_1, \ldots, w_s \in V$ and integers $d_1, \ldots, d_s$, such that the sequence A of (1) is a basis for a complementary space X_{j+1} of $\ker f^{j+1}$ inside V and the sequence of (2) is a basis of a subspace Y'_{j+1} of $\ker f^{j+1}$ with $Y'_{j+1} \cap \ker f^j = \{0\}$. Using part (a) of Lemma 1, we extend the sequence (2) to a basis

$$B = \big(f^{d_1+1}(w_1), \ldots, f^{d_s+1}(w_s), w_{s+1}, w_{s+2}, \ldots, w_t\big)$$

of a complementary space Y_{j+1} of $\ker f^j$ inside $\ker f^{j+1}$. We set $X_j = X_{j+1} + Y_{j+1}$. Then by part (b) of Lemma 1, the space X_j is a complementary space of $\ker f^j$ inside V, which, after reordering the elements of A and B, has a basis

$$\big(w_1, f(w_1), \ldots, f^{e_1}(w_1), w_2, f(w_2), \ldots, f^{e_2}(w_2), \ldots, w_t, f(w_t), \ldots, f^{e_t}(w_t)\big),$$

where $e_k = d_k + 1$ for $1 \le k \le s$ and $e_k = 0$ for $s < k \le t$. Note that this is exactly (1), with $w_1, \ldots, w_s$ replaced by $w_1, \ldots, w_t$ and $d_1, \ldots, d_s$ replaced by $e_1, \ldots, e_t$. Suppose $j > 0$, and set $Y_j' = f(Y_{j+1})$. The sequence

$$C = \left(f^{e_1+1}(w_1), \ldots, f^{e_t+1}(w_t)\right)$$

equals $f(B)$ and therefore generates Y_j'. We show that the elements in C are linearly independent. Suppose $\lambda_1, \ldots, \lambda_t \in F$ are such that

$$\sum_{k=1}^{t} \lambda_k f^{e_k+1}(w_k) = 0, \tag{3}$$

and set $x = \sum_{k=1}^{t} \lambda_k f^{e_k}(w_k) \in X_j$. Then (3) says $f(x) = 0$, so $x \in X_j \cap \ker f \subset X_j \cap \ker f^j = \{0\}$, so $x = 0$. Since the elements of B are linearly independent, we get $\lambda_1 = \cdots = \lambda_k = 0$, so the elements of C are also independent. Since Y_{j+1} is contained in $\ker f^{j+1}$, its image Y_j' is contained in $\ker f^j$. For any $y \in Y_j' \cap \ker f^{j-1}$ there is a $y' \in Y_{j+1}$ with $y = f(y')$, which satisfies $f^j(y') = f^{j-1}(y) = 0$, which implies $y' \in Y_{j+1} \cap \ker f^j = \{0\}$, so we have $y' = 0$ and hence $y = 0$. We obtain $Y_j' \cap \ker f^{j-1} = 0$. This finishes the induction argument.

The statement of the theorem follows, as for $j = 0$, the only complementary space of $\ker f^j = \ker \mathrm{id_V} = \{0\}$ is V, so we can take $(v_1, \ldots, v_n)$ to be the sequence (1) associated to $j = 0$. $\square$