Ex. 4.1. Let $\phi: \mathbb{R}^3 \to \mathbb{R}^3$ be a rotation around the line through the origin and the point $(1, 1, 1)$ by 120 degrees. Decompose $\mathbb{R}^3$ as a direct sum of two subspaces that are each stable under $\phi$.

Ex. 4.2. Consider the vector space $V = \mathbb{R}^3$ with the linear map $\phi: V \to V$ given by the matrix
\[
\begin{pmatrix}
-1 & 0 & 1 \\
-2 & -1 & 1 \\
-3 & -1 & 2
\end{pmatrix}
\]
Decompose $\mathbb{R}^3$ as a direct sum of two subspaces that are each stable under $\phi$.

Ex. 4.3. Same question for
\[
\begin{pmatrix}
0 & 1 & 1 \\
5 & -4 & -3 \\
-6 & 6 & 5
\end{pmatrix}
\]

Ex. 4.4. Consider the vector space $V = \mathbb{R}^4$ with the linear map $\phi: V \to V$ that permutes the standard basis vectors in a cycle of length 4. What is the characteristic polynomial of $\phi$? Decompose $\mathbb{R}^4$ into a direct sum of 3 subspaces that are all stable under $\phi$.

Ex. 4.5. An endomorphism $f$ of a vector space $V$ is said to be a projection if $f^2 = f$. Suppose $f$ is such a projection.

1. Show that the image of $f$ is equal to the kernel of $f - \text{id}_V$, i.e., the eigenspace $E_1$ at eigenvalue 1.
2. Show that $V$ is the direct sum of the kernel $E_0$ of $f$ and $E_1$.
3. Show that $f = f_0 \oplus f_1$ where $f_0$ is the zero-map on $E_0$ and $f_1$ is the identity map on $E_1$.

Ex. 4.6. An endomorphism $f$ of a vector space $V$ is said to be a reflection if $f^2$ is the identity on $V$. Suppose $f$ is such a reflection. Show that $V$ is the direct sum of two subspaces $U$ and $W$ for which $f = \text{id}_U \oplus (-\text{id}_W)$. 