Ex. 8.1. Let $V_1, V_2, U, W$ be vector spaces over a field $F$, and let $b: V_1 \times V_2 \to U$ be a bilinear map. Show that for each linear map $f: U \to W$ the composition $f \circ b$ is bilinear.

Ex. 8.2. Let $V, W$ be vector spaces over a field $F$. If $b: V \times V \to W$ is both bilinear and linear, show that $b$ is the zero map.

Ex. 8.3. Give an example of two vector spaces $V, W$ over a field $F$ and a bilinear map $b: V \times V \to W$ for which the image of $b$ is not a subspace of $W$.

Ex. 8.4. Let $V, W$ be two 2-dimensional subspaces of the standard $\mathbb{R}$-vector space $\mathbb{R}^3$. The restriction of the standard inner product $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ to $\mathbb{R}^3 \times W$ is a bilinear map $b: \mathbb{R}^3 \times W \to \mathbb{R}$.

1. What is the left kernel of $b$? And the right kernel?

2. Let $b': V \times W \to \mathbb{R}$ be the restriction of $b$ to $V \times W$. Show that $b'$ is degenerate if and only if the angle between $V$ and $W$ is 90°.

Ex. 8.5. Let $V, W$ be finite-dimensional vector spaces over a field $F$ and $b: V \times W \to F$ a degenerate bilinear form with left kernel $V_0$ and right kernel $W_0$. Show that $b$ induces the non-degenerate bilinear form

$$b': V/V_0 \times W/W_0 \to F, \quad (v + V_0, w + W_0) \mapsto b(v, w).$$

and conclude that $\dim(V/V_0) = \dim(W/W_0)$.

Ex. 8.6. Let $V$ be a vector space over $\mathbb{R}$, and let $b: V \times V \to \mathbb{R}$ be a symmetric bilinear map. Let the “quadratic form” associated to $b$ be the map $q: V \to \mathbb{R}$ that sends $x \in V$ to $b(x, x)$. Show that $b$ is uniquely determined by $q$.

Ex. 8.7. Let $V$ be a vector space over $\mathbb{R}$, and let $b: V \times V \to \mathbb{R}$ be a bilinear map. Show that $b$ can be uniquely written as a sum of a symmetric and a skew-symmetric bilinear form.