Density of rational points on Del Pezzo surfaces of degree one

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May 30, 2018
Surface: smooth, projective, geometrically integral scheme of finite type over a field, of dimension 2.

Surface $X$ is Del Pezzo if anticanonical divisor $-K_X$ is ample.

Degree of Del Pezzo surface $X$ is intersection number $d = K_X \cdot K_X$.

Examples:

- Intersection of two quadrics in $\mathbb{P}^4$ ($-K_S$ very ample; $d = 4$).
- A smooth cubic surface in $\mathbb{P}^3$ ($-K_S$ very ample; $d = 3$).
- Smooth double cover of $\mathbb{P}^2$, ramified over a quartic ($d = 2$). (Anticanonical map is the projection to $\mathbb{P}^2$).
**Theorem** (Segre, Manin, Kollár, Pieropan, Salgado–Testa–Várilly-Alvarado, Festi-vL).

Let $S$ be a Del Pezzo surface of degree $d \geq 2$ over a field $k$. Suppose $P \in S(k)$ is a rational point. If $d = 2$ and $k$ is infinite, then suppose, furthermore, that $P$ does not lie on four exceptional curves, nor on the ramification locus of the anticanonical map. Then $S$ is unirational over $k$.

**Theorem** (Kollár, Mella)

Let $S$ be a Del Pezzo surface of degree $d = 1$ over a field $k$ of characteristic not equal to 2. If $S$ admits a (certain?) conic bundle structure, then $S$ is unirational.

**Remark.** When these are minimal, Picard number $\rho(S) = 2$.

**Question 1.** Is there a DP1 with $\rho = 1$ that is unirational?

**Question 2.** Is there a DP1 with $\rho = 1$ that is **not** unirational?
**Conjecture** (Batyrev–Manin–Peyre). Suppose that $S$ is a Del Pezzo surface of degree 1 over a number field $k$ with a rational point and Picard number $\rho$. Then there is a nonzero constant $c$ such that for every small enough nonempty open subset $U \subset S$ we have

$$\# \{ P \in U(k) : H_{-K}(P) \leq B \} \sim cB(\log B)^{\rho-1}$$

as $B \to \infty$.

Nonzeroness comes from Peyre’s description of $c$ or Colliot-Thélène’s conjecture that Brauer–Manin is the only obstruction to weak approximation on rational varieties.

**Conclusion.** There should be lots of rational points!
Every Del Pezzo surface $S/k$ of degree $d = 1$ is isomorphic to a smooth sextic in $\mathbb{P}(2, 3, 1, 1)$, with coordinates $x, y, z, w$, given by

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

with $a_i \in k[z, w]$ homogeneous of degree $i$. (And vice versa.)

Linear system $|-K_S|$ induces rational map $S \dashrightarrow \mathbb{P}^1(z, w)$.

Unique base point $O = [1 : 1 : 0 : 0]$.

Curves in $|-K_S|$ are fibers of $\text{Bl}_O(S) \rightarrow \mathbb{P}^1$ (anticanonical fiber).

Almost all are elliptic fibers, all are geometrically integral.
Theorem (Várilly-Alvarado). Let $A, B$ be nonzero integers, and let $S$ be the Del Pezzo surface of degree 1 over $\mathbb{Q}$ given by

$$y^2 = x^3 + Az^6 + Bw^6.$$

Assume that Tate–Shafarevich groups of elliptic curves over $\mathbb{Q}$ with $j$-invariant 0 are finite. If $3AB$ is not a square, or if $A$ and $B$ are relatively prime and $9 \nmid AB$, then $S(\mathbb{Q})$ is Zariski dense in $S$.

Question. What about $y^2 = x^3 + 243z^6 + 16w^6$?
Theorem (Salgado–vL). Let $S \subset \mathbb{P}(2, 3, 1, 1)$ be a Del Pezzo surface of degree 1 over a number field $k$. Suppose $F$ is a smooth anticanonical fiber with a point $P \in F(k)$ of order $> 2$ that does not lie on six exceptional curves. Set $U = \mathbb{P}(2, 3, 1, 1) \setminus Z(z, w)$ and let $C$ denote the curve of those sections of the projection $U \to \mathbb{P}^1(z, w)$ that meet $S$ at the point $P$ with multiplicity at least 5. If $\#C(k) = \infty$, then $S(k)$ is Zariski dense.

Remark. If the order of $P$ is 3, then $\#C(k) = \infty$ for free.
For Várilly-Alvarado’s example

$$y^2 = x^3 + 243z^6 + 16w^6$$

there is a 3-torsion point $[0 : 4 : 0 : 1]$, but it lies on nine exceptional curves...

The coefficients of the curve $C$ associated to the point $[-63 : 14 : 1 : 5]$ are too large to find points...
Elkies' proof for Várilly-Alvarado's example. Take affine part

\[ y^2 = x^3 + 243 + 16t^6, \]

and set \( y = v + 4t^3 \) to obtain

\[ v^2 + 8t^3v = x^3 + 243. \]

Now projection onto \( v \)-line gives fibration into cubics. The point \((t, x, y) = (5, -63, 14)\) gives a cubic with infinitely many rational points, most of which have infinite order on their anticanonical fiber. Done!

Elkies: When I tried to generalise this construction, it turned out I needed a 3-torsion point. Maybe it can be done for all rational torsion points (on their anticanonical fiber).
Theorem (Bulthuis-vL). Let $S$ be a Del Pezzo surface of degree 1 over a number field $k$. Suppose $F$ is a smooth anticanonical fiber with a point $P \in F(k)$ of finite order $n > 1$. Then the linear system
\[ | - nK_S - nP| = \{ D \in | - nK_S| : \mu_P(D) \geq n \} \]
induces a fibration $\varphi : \text{Bl}_P(S) \to \mathbb{P}^1$ of curves of genus 1. If
1) some irreducible fiber of $\varphi$ has $\infty$ many $k$-rational points, or
2) there is a $Q \in S(k) \setminus \{P\}$ such that the fiber $G_Q = \varphi^{-1}(\varphi(Q))$ is smooth, and on the elliptic curve $(G_Q, Q)$, the sum of the points above $P$ has infinite order,
then $S(k)$ is Zariski dense.

Moreover, there is a nonempty Zariski open subset $U \subset S$ such that every $Q \in U(k)$ satisfies the conditions of 2).

**Consequence.** Suppose $S$ has a point that has finite order on its anticanonical fiber. If $S(k)$ is Zariski dense in $S$, then there is an easy proof that this is the case.
Set $\pi: S' = \text{Bl}_P(S) \to S$, except curve $E$, strict transf. $F'$ of $F$.

1. $| - nK_S - nP| \leftrightarrow | - n\pi^*K_S - nE| = |n\pi^*F - nE| = |nF'|.$
2. $\dim |mF'| = 0$ for $0 \leq m < n$, and $\dim |nF'| = 1$.
3. Base locus of $|nF'|$ is empty: 1-dim’l components are $F'$, contradicting 2. No base points as $F'^2 = (\pi^*F - E)^2 = 0$.
4. Bertini Theorem: almost all curves in $|nF'|$ smooth.
5. All curves connected: Stein factorisation gives $S' \to \mathbb{P}^1$ with fibers $H$ with $aH \sim nF'$ for some $a$, contradicting 2.
6. Almost all curves geometrically integral.
7. Genus of smooth $D \in |nF'|$ is 1: we have $-K_{S'} \sim F'$, so $2g(D) - 2 = D(D + K_{S'}) = D(D - F') = 0$.
8. Fibers of $\varphi$ are not in torsion locus of anticanonical fibration: torsion-locus does not self intersect in smooth fibers.
9. Equivalent condition on $Q$: the divisor $n(Q) - \sum_{P'|P}(P') \in \text{Jac}(G_Q)$ is not torsion.
To find \( \dim |mF'| = \dim H^0(S', \mathcal{O}_{S'}(mF')) - 1 \), we consider the embedding \( \iota : F' \hookrightarrow S' \) and exact sequence (idea Adam Logan)

\[
0 \to \mathcal{O}_{S'}(-F') \to \mathcal{O}_{S'} \to \iota_* \mathcal{O}_{F'} \to 0
\]

of sheaves on \( S' \). Twisting by \( \mathcal{O}_{S'}(mF') \) gives

\[
0 \to \mathcal{O}_{S'}((m - 1)F') \to \mathcal{O}_{S'}(mF') \to \iota_* \mathcal{O}_{F'} \otimes \mathcal{O}_{S'}(mF') \to 0.
\]

Let \( F_\infty \) be an other anticanonical fiber and \( F'_\infty = \pi^* F_\infty \) its strict transform. Then \( F' \sim F'_\infty - E \), so

\[
\iota_* \mathcal{O}_{F'} \otimes \mathcal{O}_{S'}(mF') \cong \iota_* \mathcal{O}_{F'} \otimes \mathcal{O}_{S'}(mF'_\infty - mE) \cong \iota_* \mathcal{O}_{F'}(mO - mP).
\]

We obtain

\[
0 \to \mathcal{O}_{S'}((m - 1)F') \to \mathcal{O}_{S'}(mF') \to \iota_* \mathcal{O}_{F'}(mO - mP) \to 0.
\]

Now we take the associated long exact sequence.
\[ 0 \to H^0(S', \mathcal{O}((m - 1)F')) \to H^0(S', \mathcal{O}(mF')) \to H^0(F', mO - mP) \to H^1(S', \mathcal{O}((m - 1)F')) \to H^1(S', \mathcal{O}(mF')) \to H^1(F', mO - mP) \]

Using (for \( i \in \{0, 1\} \))

\[
\dim H^i(F', mO - mP) = \begin{cases} 
1 & \text{if } n|m \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
\dim H^i(S', \mathcal{O}_{S'}) = \begin{cases} 
1 & \text{if } i = 0 \\
0 & \text{if } i = 1
\end{cases},
\]

we find, by induction, that (for \( i \in \{0, 1\} \) and \( 0 < m < n \))

\[
\dim H^i(S', \mathcal{O}_{S'}(mF')) = \begin{cases} 
1 & \text{if } i = 0 \\
0 & \text{if } i = 1
\end{cases}.
\]

Finally, for \( m = n \), we obtain

\[
\dim H^i(S', \mathcal{O}_{S'}(nF')) = 2.
\]
Thanks!