Computing Néron-Severi groups

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Setting

- $k$ a finitely generated field.
- $X$ a nice $k$-variety (smooth, projective, geometrically integral).
- $k^s$ a separable closure of $k$ with Galois group $\Gamma = \text{Gal}(k^s/k)$.
- $X^s = X \times_k k^s$.
- Picard group $\text{Pic} X \subset (\text{Pic} X^s)^\Gamma$.
- $\text{Pic}^0(X)$ subgroup of classes algebraically equivalent to 0.
- Néron-Severi group $\text{NS}(X) = \text{Pic} X / \text{Pic}^0(X) \subset \text{NS}(X^s)^\Gamma$.

Goal: Compute $\text{NS}(X^s)$ (or its rank).
Special cases

- Elliptic fibrations: map $\text{NS}$ to the Mordell-Weil group.
- Fibrations into abelian varieties.
- If a finite group $G$ acts on $Y$ and $X = Y/G$, then

$$\text{NS}(X^s) \otimes \mathbb{Q} \to (\text{NS}(Y^s) \otimes \mathbb{Q})^G$$

is an isomorphism.

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Application (Shioda):

Delsarte surfaces (given by tetranomials in $\mathbb{P}^3$) are quotients of Fermat surfaces.
K3 surfaces of degree 2

Theorem (Hassett, Kresch, Tschinkel)

There is an algorithm that takes as input a K3 surface $X$ of degree 2 over a number field, and returns $\text{Pic } X^s = \text{NS } X^s$.

Method: Kuga-Satake correspondence.
Ingredients include: abelian variety of dimension $2^{19}$. 
Tate conjecture(s)

- Fix $0 \leq p \leq \dim X$ and prime $\ell \neq \text{char } k$.
- $\mathcal{Z}^p(X)$ is group of codimension-$p$ cycles on $X$.
- $V^{2p} = H^{2p}_{et}(X^s, \mathbb{Q}_\ell(p))$.
- $V^{\text{tate}} \subset V^{2p}$ is set of Tate classes (each fixed by some finite-index open subgroup $G \subset \Gamma$).

Conjecture ($T^p(X, \ell)$)

The cycle class map $\mathcal{Z}^p(X^s) \otimes \mathbb{Q}_\ell \to V^{\text{tate}}$ is surjective.
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Conjecture ($E^p(X, \ell)$)

An element in $\mathcal{Z}^p(X^s, \ell)$ is numerically equivalent to 0 if and only if it maps to 0 in $V^{2p}$.

Remark. $E^p(X, \ell)$ holds for $p = 1$. 
Algorithms for general $p$

- $\text{Num}^p(X)$ is group of codimension-$p$ cycle classes up to numerical equivalence.

- Assuming $E^p(X, \ell)$, the map $\text{Num}^p(X, \ell) \otimes \mathbb{Q}_\ell \hookrightarrow V^{\text{tate}}$ is an injection that is an isomorphism if and only if $T^p(X, \ell)$ holds.

- For $p = 1$ we have $\text{Num}^1(X) \cong \text{NS}(X) / \text{NS}(X)_{\text{tors}}$ and an injection $\text{NS}(X) \otimes \mathbb{Q}_\ell \rightarrow V^{\text{tate}}$. 

Strategy for bounding $\text{rk Num}^p(X)$.
1. List cycles to find lower bound (also cycles to intersect with).
2. Bound $\text{dim } \mathbb{Q}_\ell V^{\text{tate}}$ from above for upper bound.

Trivial upper bound: Betti number $b_2^p$. 

Problem with computing $V^{\text{tate}} \subset V_2^p = H_2^p_{\text{et}}(X_s, \mathbb{Q}_\ell(1))$ is that $\mathbb{Q}_\ell$ requires infinite precision and $k_s$ may not be finitely generated.
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Hypothesis. Can compute $T_{\ell^n}^i = H_{et}^i(X^s, \mathbb{Z}/\ell^n\mathbb{Z})$ as $\Gamma$-module.
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Theorem (Poonen, Testa, vL)

Assume the hypothesis. Then there is an algorithm that takes as input $(k, p, X, \ell)$ as before, such that, assuming $E^p(X, \ell)$, the algorithm terminates if and only if $T^p(X, \ell)$ holds, and if the algorithm terminates, it returns $\text{rk Num}^p(X^s)$. 

Sketch of proof of upper bound for $r = \dim V_{\text{tate}}$.

1. Extend $k$ so that $\Gamma$ acts trivially on $T_{\ell^{\prime}}(p)$ with $\ell^{\prime} = \ell$ for $\ell > 2$ and $\ell^{\prime} = 4$ for $\ell = 2$.

2. $\Gamma$ acts trivially on $M/\mathcal{M}_{\text{tors}}$ with $M = H^2_{et}(X^s, \mathbb{Z}/\ell\mathbb{Z}(p))_{\text{tate}}$ (Minkovski’s Lemma on finite-order elements in $\text{GL}_n(\mathbb{Z}/\ell\mathbb{Z})$).

3. Compute $t$ such that $\ell^t$ kills $H^2_{et}(X^s, \mathbb{Z}/\ell\mathbb{Z}(p))_{\text{tors}}$ (Wittenberg).

4. $\ell^{r(n-t)} \leq \#(\mathcal{M}/\ell\mathcal{M}) \leq \#(\mathcal{M}/\ell^n\mathcal{M})_{\Gamma} = O(\ell^{rn})$.

5. Sequence $\lfloor \log \#T_{\ell^n}(p)_{\Gamma} \log \ell^n - t \rfloor$ (with $n \geq 1$) has minimum $r$. 
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5. Sequence $\left\lfloor \frac{\log \# T^{2p}_{\ell^n}(p)^\Gamma}{\log \ell^{n-t}} \right\rfloor$ (with $n \geq 1$) has minimum $r$. 
Finite fields

Suppose $k$ is finite. Let $V_\mu \subset V^{2p} = H^{2p}_{et}(X^s, \mathbb{Q}_\ell(p))$ be the largest subspace on which all eigenvalues of Frobenius are roots of unity.

$$\text{Num}^p(X^s) \otimes \mathbb{Q}_\ell \to V^{\text{tate}} \subset V_\mu$$
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Theorem

Assuming $E^p(X, \ell)$, the following are equivalent.

1. $\text{rk} \text{Num}^p(X^s) = \dim V^{\text{tate}}$.
2. Conjecture $T^p(X, \ell)$ holds.
3. $\text{rk} \text{Num}^p(X^s) = \dim V^{\text{tate}} = \dim V_{\mu}$. 

Proof. 1 $\iff$ 2. Under $E^p(X, \ell)$, the first map is injective, so it is surjective if and only if 1 holds. 2 $\Rightarrow$ 3. $V^{\text{tate}} = V_{\mu}$ follows as $E^p(X, \ell)$ and $T^p(X, \ell)$ together imply that Frobenius acts semi-simple on $V_{\mu}$. 3 $\Rightarrow$ 1. Obvious.
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Proof. $1 \Leftrightarrow 2$. Under $E^p(X, \ell)$, the first map is injective, so it is surjective if and only if 1 holds.

$2 \Rightarrow 3$. $V^{\text{tate}} = V_\mu$ follows as $E^p(X, \ell)$ and $T^p(X, \ell)$ together imply that Frobenius acts semi-simple on $V_\mu$. $3 \Rightarrow 1$. Obvious.
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Theorem

There is an algorithm that takes as input \((k, p, X, \ell)\), with \(k\) a finite field, and that, assuming \(E^p(X, \ell)\), terminates if and only if \(T^p(X, \ell)\) holds, and if it terminates, it returns \(\text{rk Num}^p X^s\).
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Proof. By searching for cycles, we get a lower bound for \(\text{rk Num}^p X^s\) that eventually is sharp. To verify that it is, it suffices to compute \(\dim V_\mu\). Say \(k = \mathbb{F}_q\). The degree of the zeta-function

\[
Z_X(T) = \prod_{i=0}^{2\dim X} \left( \det (1 - T \cdot \text{Frob}^*| H^i_{\text{et}}(X^s, \mathbb{Q}_\ell)) \right)^{(-1)^{i+1}}
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is bounded by the sum of Betti numbers: computable bound \(B\).
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is bounded by the sum of Betti numbers: computable bound \(B\). Computing \(#X(\mathbb{F}_{q^n})\) for \(n = 1, \ldots, 2B\) gives enough information to determine \(Z_X(T)\). Then \(\dim V_\mu\) equals the number of poles of \(Z_X(T)\) that are roots of unity times \(q^{-p}\).
Finite fields

Example. Let $X \subset \mathbb{P}^3$ over $\mathbb{F}_2$ be given by $\det M = 0$ with $M =$

$$
\begin{pmatrix}
    x_0 & x_1 & x_2 + x_3 & x_1 + x_2 & x_2 + x_3 \\
    x_1 & x_2 + x_3 & x_0 + x_1 + x_2 + x_3 & x_0 + x_1 & x_2 \\
    x_0 + x_2 & x_0 + x_1 + x_2 + x_3 & x_0 + x_1 & x_2 \\
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$\#X(\mathbb{F}_{2^n}) = 1 + 2^n t_n + 2^{2n}$

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\( f_\Phi = \) palindromic or antipalindromic

\[
f_\Phi = t^{22} - \frac{1}{2} t^{21} - t^{20} - \frac{1}{2} t^{19} + t^{18} - \frac{1}{2} t^{15} + t^{14} + \frac{1}{2} t^{13} - 2 t^{11} + \ldots
\]

\[
= (t - 1)^2(t^{20} + \frac{3}{2} t^{19} + t^{18} - \frac{1}{2} t^{13} + t^{11} + 2 t^{10} + \ldots).
\]

Conclusion. We have \( \text{rk} \text{NS}(X^s) = 2 \).
Surfaces over global fields

- Global field $K$, discrete valuation ring $R \subset K$, residue field $k$.
- $X$ a nice surface over $K$, integral model $\mathcal{X}$ over $R$ with good reduction.

\[
\text{NS}(X^s) \otimes \mathbb{Q}_\ell \hookrightarrow \text{NS}(\mathcal{X}_{ks}) \otimes \mathbb{Q}_\ell
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\text{rk } \text{NS}(X^s) \leq \text{rk } \text{NS}(\mathcal{X}_{ks})
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$$\text{rk NS}(X^s) \leq \text{rk NS}(\mathcal{X}_{ks})$$

Problem.
If $T^1(\mathcal{X}_{ks}, \ell)$ holds, then $\text{rk NS}(\mathcal{X}_{ks}) \equiv b_2(X) \pmod{2}$.

Proof.
The roots of $f_\Phi$ that are not roots of unity come in conjugate pairs.

Question.
How to ever prove $\text{rk NS}(X^s) = 1$ for a K3 surface over $K = \mathbb{Q}$?
The injection
\[ \text{Num}^1(X^s) \hookrightarrow \text{Num}^1(X_{ks}) \]
respects the intersection pairing.

**Lemma.** If \( \Lambda' \) is a sublattice of finite index of \( \Lambda \), then we have
\[ \text{disc } \Lambda' = [\Lambda : \Lambda']^2 \text{disc } \Lambda. \]
Hence, \( \text{disc } \Lambda = \text{disc } \Lambda' \) in \( \mathbb{Q}^*/(\mathbb{Q}^*)^2 \).
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Hence, \(\text{disc} \Lambda = \text{disc} \Lambda'\) in \(\mathbb{Q}^*/(\mathbb{Q}^*)^2\).

**Corollary (vL).** If \(v, w\) are two places of good reduction with
1) \(\text{rk} \text{Num}^1(X_{k(v)s}) = r = \text{rk} \text{Num}^1(X_{k(w)s})\), and
2) \(\text{disc} \text{Num}^1(X_{k(v)s}) \neq \text{disc} \text{Num}^1(X_{k(w)s})\) in \(\mathbb{Q}^*/(\mathbb{Q}^*)^2\),
then \(\text{rk} \text{Num}^1(X^s) < r\).

If \(\text{rk} \text{Num}^1(X^s) = r - 1\), then this equality is verifiable.
Example. Let $X \subset \mathbb{P}_\mathbb{Q}^3$ be given by

$$wf = 3pq - 2zg$$

with $f \in \mathbb{Z}[x, y, z, w]$ and $g, p, q \in \mathbb{Z}[x, y, z]$ equal to

$$f = x^3 - x^2y - x^2z + x^2w - xy^2 - xyz + 2xyw + xz^2 + 2xzw + y^3 + y^2z - y^2w + yz^2 + yzw - yw^2 + z^2w + zw^2 + 2w^3,$$

$$g = xy^2 + xyz - xz^2 - yz^2 + z^3,$$

$$p = z^2 + xy + yz,$$

$$q = z^2 + xy.$$

Then $\text{rk NS}(X^s) = 1$. 
Example. Let $X \subset \mathbb{P}^3_Q$ be given by

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with ... Then $\text{rk NS}(X^s) = 1$.

Proof. Two primes of good reduction: 2 and 3. For both we obtain $\dim V_\mu = 2$ as before. Reduction $X_{\mathbb{F}_2}$ contains conic $C$ given by $w = p = 0$. Reduction $X_{\mathbb{F}_3}$ contains line $L$ given by $w = z = 0$. 
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Proof. Two primes of good reduction: 2 and 3. For both we obtain $\dim V_\mu = 2$ as before. Reduction $X_{F_2}$ contains conic $C$ given by $w = p = 0$. Reduction $X_{F_3}$ contains line $L$ given by $w = z = 0$. Then $\text{Num}^1(X_{F_2})$ and $\text{Num}^1(X_{F_3})$ contain finite-index sublattices

$$\begin{pmatrix} 4 & 2 \\ 2 & -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}$$

with discriminants $-12$ and $-9$ in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$. Not the same, so $\text{rk NS}(X^s) = 1$. 
Extension by Kloosterman

- $k = \mathbb{F}_q$.
- $X/k$ a nice surface.
- $\Phi = \text{Frob}^*| H^2(X^s, \mathbb{Q}_\ell)$.
- $f_\Phi(T) = \det(1 - T \cdot \Phi)$.
- $\rho = \text{rk Num}^1(X)$ and $\Delta = \text{disc Num}^1(X)$.
- $b_2 = b_2(X)$ and $\alpha = \chi(X, \mathcal{O}_X) - 1 - \dim \text{Pic}^0(X)$.

Conjecture (Artin–Tate).

$$\lim_{T \to q^{-1}} \frac{f_\Phi(T)}{(1 - qT)^\rho} = \frac{(-1)^{\rho-1} \cdot \# \text{Br}_X \cdot \Delta}{q^{\alpha} \left(\# \text{NS}(X)_{\text{tors}}\right)^2}.$$
Extension by Kloosterman

- $k = \mathbb{F}_q$.
- $X/k$ a nice surface.
- $\Phi = \text{Frob}^*|H^2(X^s, \mathbb{Q}_\ell)$.
- $f_\Phi(T) = \det(1 - T \cdot \Phi)$.
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Facts.

$T^1(X, \ell) \Rightarrow$ Artin–Tate.

$T^1(X, \ell) \Rightarrow \# \text{Br} X \in (\mathbb{Q}^*)^2$ (Liu–Lorenzini–Raynaud).

Conclusion. We may compute $\Delta \in \mathbb{Q}^*/\mathbb{Q}^{*2}$, assuming $T^1(X, \ell)$. 
Application

Theorem (Kloosterman)

The elliptic K3 surface $\pi : X \to \mathbb{P}^1$ over $\overline{\mathbb{Q}}$ given by

$$y^2 = x^3 + 2(t^8 + 14t^4 + 1)x + 4t^2(t^8 + 6t^4 + 1)$$

has $\text{rk NS}(X) = 17$ and Mordell-Weil rank 15.
Extension by Elsenhans-Jahnel, I

Main idea. If you consider Galois action, you may not need
\[ r = \text{rk } \text{Num}^1 (X^s) + 1. \]
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Example. (Elsenhans, Jahnel)
Let \( X : w^2 = f(x, y, z) \) be a K3 surface of degree 2 over \( \mathbb{Q} \) with
\[
f \equiv y^6 + x^4y^2 - 2x^2y^4 + 2x^5z + 3xz^5 + z^6 \quad (\text{mod } 5)
\]
and
\[
f \equiv 2x^6 + x^4y^2 + 2x^3y^2z + x^2y^2z^2 + x^2yz^3 + 2x^2z^4 + xy^4z
\plus xy^3z^2 + xy^2z^3 + 2xz^5 + 2y^6 + y^4z^2 + y^3z^3 \quad (\text{mod } 3).
\]
Then \( \text{rk} \text{NS}(X^s) = 1. \).
Extension by Elsenhans-Jahnel, I

Let $L$ denote the pull-back of a line in $\mathbb{P}^2(x, y, z)$.

The characteristic polynomial of Frobenius acting on the space

$$(\text{NS} \times_{\mathbb{F}_3} \otimes \mathbb{Q})/\langle L \rangle$$

equals $(t - 1)(t^2 + t + 1)$, so only finitely many Galois-invariant subspaces of $\text{NS} \times_{\mathbb{F}_3} \otimes \mathbb{Q}$ containing $L$; dimensions are $1, 2, 3, 4$. 
Extension by Elsenhans-Jahnel, I

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The characteristic polynomial of Frobenius acting on the space $(\text{NS } X_{\overline{F}_5} \otimes \mathbb{Q})/\langle L \rangle$
equals (t - 1)\Phi_5(t)\Phi_{15}(t), where $\Phi_n$ denotes the $n$-th cyclotomic polynomial. So only finitely many Galois-invariant subspaces of $\text{NS } X_{\overline{F}_5} \otimes \mathbb{Q}$ containing $L$; dimensions are $1, 2, 5, 6, 9, 10, 13, 14$. 

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Only common dimensions are $1$ and $2$. Comparing discriminants up to squares of the subspaces of dimension $2$ yields $\text{rk } \text{NS}(X^s) = 1$. 
Extension by Elsenhans-Jahnel, II

- \( p \neq 2 \) prime.
- \( X \) a scheme that is proper and flat over \( \mathbb{Z} \).

**Theorem** (Elsenhans-Jahnel). If the special fiber \( X_p \) is nonsingular, then the cokernel of the specialization homomorphism

\[
\text{sp}_Q : \text{Pic}(X_\mathbb{Q}) \to \text{Pic}(X_{\overline{p}})
\]

is torsion-free.
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Theorem (Elsenhans-Jahnel). If the special fiber $X_p$ is nonsingular, then the cokernel of the specialization homomorphism

$$\text{sp}_Q : \text{Pic}(X_Q) \to \text{Pic}(X_p)$$

is torsion-free.

Let $X$ be a double cover of $\mathbb{P}^2$, ramified over a smooth plane sextic $C$. Let $p, p'$ denote two odd primes of good reduction. Assume that there is a tritangent line $\ell$ to the curve $C_p$. Suppose $\text{Pic}(X^s_p)$ has rank 2 and is generated by the components in the pull-back of $\ell$. If there are no tritangent lines to $C_{p'}$, then $\text{rk} \text{Pic}(X^s) = 1$. 
“It works” by Charles

Question.
1) Given a nice surface $X$ over a number field $k$, is there always a prime $p$ of good reduction with $\text{rk Num}^1(X_p) \leq \text{rk Num}^1(X^s) + 1$?
2) Are there two so that the discriminant trick works?
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Consequence (Charles).
There is an algorithm with input a K3 surface $X$ over a number field that either returns $\text{rk NS}(X^s)$ or does not terminate.
If $X \times X$ satisfies the Hodge conjecture for codimension 2 cycles, then the algorithm applied to $X$ terminates.
Saturation

Theorem (Poonen, Testa, vL)

There is an algorithm that takes $k$, $X$, and a finite set $\mathcal{D}$ of divisors as input, and computes the saturation inside $\text{NS}(X^s)$ of the $\Gamma$-submodule generated by the classes of divisors in $\mathcal{D}$.

Method. Hilbert scheme computations.
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**Goal.** Given a surface $X$ over a global field $K$ and a sublattice $G \subset \text{Num}^1(X^s)$, show that $G$ is primitive.
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If not primitive, then \( G \) has nontrivial index in its saturation \( \tilde{G} \), so there is a prime \( r \mid [\tilde{G} : G] \) with \( r^2 \mid [\tilde{G} : G]^2 \cdot \text{disc} \tilde{G} = \text{disc} G \). Then \( G \otimes \mathbb{F}_r \to \text{Num}^1(X^s) \otimes \mathbb{F}_r \) is not injective.
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For all primes $p$ of good reduction and $H \subset \text{Num}^1(X_p^s)$ the map

$$\text{Num}^1(X^s) \to \text{Hom}(H, \mathbb{Z})$$

induces a non-injective composition

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Sufficient for primitivity. Find for each $r$ with $r^2 | \text{disc} G$ a prime $p$ and a subgroup $H \subset \text{Num}^1(X^s_p)$ for which the composition is injective (linear algebra over $\mathbb{F}_r$).
Application

Theorem (Mizukami \((m = 4)\), Schütt–Shioda–vL \((m \leq 100)\))

For any integer \(1 \leq m \leq 100\) the Néron-Severi group of the Fermat surface \(S_m \subset \mathbb{P}^3\) over \(\mathbb{C}\) given by

\[x^m + y^m + z^m + w^m = 0\]

is generated by the lines on \(S_m\) if and only if \(m \leq 4\) or \((m, 6) = 1\).