Manin Conjectures for K3 surfaces

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A K3 surface

with 2 singular curves of genus 0 and several rational points
Question 1. Is there a K3 surface, defined over a number field, such that its set of rational points is neither empty, nor dense?
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Examples of K3 surfaces are smooth quartic surfaces in $\mathbb{P}^3$.

Question 2 (Swinnerton-Dyer).

Does $X \subset \mathbb{P}^3$ given by

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have more than 2 rational points?

**Answer (Elsenhans, Jahnel, 2004):**

$$1484801^4 + 2 \cdot 1203120^4 = 1169407^4 + 4 \cdot 1157520^4$$
We will look at the growth of the number of rational points of bounded height. Consider a surface $X/K$, choose a height $H$, and set

$$N_U(B) = \# \{ x \in U(K) : H(x) \leq B \}.$$

**Conjecture 1 (Batyrev, Manin).** Let $X$ be a smooth, geometrically integral, projective variety over a number field $K$, and let $D$ be a hyperplane section. Assume that the canonical sheaf $K_X$ satisfies $-K_X = aD$ for some $a > 0$. Then there exists a finite field extension $L$, a constant $C$, and an open subset $U \subset X$, such that with $b = \text{rk} \text{Pic}(X_L)$ we have

$$N_{UL}(B) \approx CB^a(\log B)^{b-1}.$$
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What if $a = 0$, in particular, if $X$ is K3?

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Define the height-zeta function

\[ Z(U, s) = \sum_{x \in U(K)} H(x)^{-s}. \]

From

\[ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^s \frac{ds}{s} = \begin{cases} 
1 & \text{if } x > 1 \\
\frac{1}{2} & \text{if } x = 1 \\
0 & \text{if } x < 1 
\end{cases} \quad (c > 0) \]

we get

\[ N(U, x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z(U, s) x^s \frac{ds}{s} \quad (c >> 0) \]
Assuming $Z(U, s)$ is analytic on $\Re(s) > a - \epsilon$, except for a pole of order $b$ at $a$, we can write

\[
N(U, x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z(U, s) x^s \frac{ds}{s} \quad (c >> 0)
\]

\[
= \res_{s=a} [Z(U, s)s^{-1} \exp(s \log x)] + \frac{1}{2\pi i} \int_{a-\epsilon-i\infty}^{a-\epsilon+i\infty} Z(U, s) x^s \frac{ds}{s}.
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The main term is

$$x^a p(\log x)$$

for some polynomial $p$ of degree $\begin{cases} b - 1 & \text{if } a \neq 0, \\ b & \text{if } a = 0. \end{cases}$
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**For $X$ a K3 surface:** $N(U, B) \sim C(\log B)^{\text{rk} \text{Pic} X}$
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Not if $X$ admits an elliptic fibration (in particular, if $\text{rk Pic } X \geq 5$).
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More problems:
- infinitely many curves of genus 0 or 1,
- infinitely many automorphisms (cf. A. Baragar),
- Swinnerton-Dyer’s surface has very slow growth,
- which height to use?
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- infinitely many curves of genus 0 or 1,
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- which height to use?

We will run experiments on quartic surfaces, comparing exponents, and the main coefficient $C$ to a relatively naive heuristic coefficient $\tilde{C}$ (part of Peyre’s constant for del Pezzo).

There are some subtle rational numbers that $C$ and $\tilde{C}$ may be off by.
\( \bar{C} \sim 6.24 \)

Line:
\[ \bar{C}(\log B) - 9 \]

\( \rho(X_{\overline{Q}}) = 1 \)

\# Aut \( X_{\overline{Q}} < \infty \)

No elliptic fibration

\[
f = x^3w - 3x^2y^2 - x^2yw - x^2zw + x^2w^2 \\
-xy^2z - xy^2w - 4xyz^2 - xyzw + 2xyw^2 - 2xz^3 \\
+ xz^2w + 2xzw^2 + y^3w + y^2zw - y^2w^2 - 5yz^3 \\
+ yz^2w + yzw^2 - yw^3 - z^4 + z^2w^2 + zw^3 + 2w^4
\]
\[
\tilde{C} \sim 0.57 \\
\text{line: } \tilde{C}(\log B) \\
\rho(X) = 1 \\
\rho(X_{\overline{Q}}) = 20 \\
\# \text{Aut } X_{\overline{Q}} = \infty \\
\text{no elliptic fibration over } \mathbb{Q}
\]

\[
f = x^4 + 2y^4 - z^4 - 4w^4
\]
\[ \tilde{C} \sim 7.8 \]

line: \[ \frac{3}{4} \tilde{C}(\log B) - 10 \]

\[ \rho(X_{\mathbb{Q}}) = 1 \]

\[ \# \text{Aut} \, X_{\mathbb{Q}} < \infty \]

no elliptic fibration

\[ f = x^3w - 3x^2y^2 + 4x^2yz - x^2z^2 - x^2z^2 + x^2zw - xy^2z - xyz^2 + xw^3 + y^3w + y^2z^2 + z^3w \]
Consider $X$ given by

$$f + 6m(w^4 + 2x^4 - (y + w)^4 - 4z^4)$$

for $m \in \{\pm1, \pm2, \pm3\}$ with $f$ as on the previous page.

Then $X$ has geometric Picard number 1.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$\bar{C}$</th>
<th>$\bar{C} \cdot \log(12000)$</th>
<th>$#{x : H(x) \leq 12000}$</th>
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<tr>
<td>0</td>
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<td>73</td>
<td>46</td>
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<td>5.4</td>
<td>9</td>
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<td>3.7</td>
<td>2</td>
</tr>
<tr>
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<td>6</td>
</tr>
<tr>
<td>3</td>
<td>0.18</td>
<td>1.7</td>
<td>4</td>
</tr>
<tr>
<td>−3</td>
<td>0.26</td>
<td>2.4</td>
<td>4</td>
</tr>
</tbody>
</table>
\[ \tilde{C} \sim 6.66 \]

line: \( \frac{1}{2} \tilde{C}(\log B) \)

\( \rho(X) = 1 \)

\( \rho(X_{\overline{\mathbb{Q}}}) = 2 \) (over \( \mathbb{Q}(\sqrt{3}) \))

Aut \( \overline{X_{\overline{\mathbb{Q}}}} = \{1\} \)

no elliptic fibration

\[
\begin{align*}
f &= x^3w - x^2y^2 - 2x^2yz \\
   &\quad - x^2z^2 + x^2zw + xy^2z + xyz^2 \\
   &\quad + xw^3 + y^3w - y^2z^2 + z^3w
\end{align*}
\]
\[ \mathcal{C} \sim 0.96 \]
parabola: \[ 1.1(\log B)^2 \]
\[ \rho(X) = 2 \]
\[ \rho(X_{\overline{Q}}) = 2 \]
\[ \text{Aut } X_{\overline{Q}} = \{1\} \]
no elliptic fibration

\[ f = 3x^3w - x^2y^2 - 2x^2yz - x^2z^2 - 4xy^2z + 3xy^2w - 4xyz^2 + 4xyw^2 + 3xz^2w + 4xw^3 + y^3w - 3y^2z^2 + 4y^2w^2 + 2z^3w + w^4 \]
\[ \log(\# \text{ pt's}) \]

\[ \bar{C} \sim 0.96 \]

\[ \text{line: } 2(\log \log B) + \log 1.1 \]

\[ \rho(X) = 2 \]

\[ \rho(X_{\overline{Q}}) = 2 \]

\[ \text{Aut } X_{\overline{Q}} = \{1\} \]

\[ \text{no elliptic fibration} \]

\[ f = 3x^3w - x^2y^2 - 2x^2yz \]

\[ -x^2z^2 - 4xy^2z + 3xy^2w - 4xyz^2 \]

\[ +4xyw^2 + 3xz^2 + 4xw^3 + y^3w \]

\[ -3y^2z^2 + 4y^2w^2 + 2z^3w + w^4 \]
$\tilde{C} \sim 0.53$

Parabola: $\frac{3}{2} \tilde{C}((\log B) - 2)^2 + 2$

$\rho(X) = 2$

$\rho(X_{\overline{Q}}) = 2$

$\text{Aut } X_{\overline{Q}} = \{1\}$

no elliptic fibration

$f = -6x^3y - 6x^3z - 3x^3w - x^2y^2 + 16x^2yz + 35x^2z^2 + 2xy^2z + 3xy^2w + 8xyz^2 - 2xyw^2 + 6xz^3 + 3zw^2 - 2xw^3 + y^3w + 3y^2z^2 + 4y^2w^2 + 6yz^3 + 2z^3w + 6z^2w^2 - 6zw^2 + w^4$
\[ \tilde{C} \sim 0.53 \]

line: \[ 2(\log \log B) - 0.75 \]

\[ \rho(X) = 2 \]

\[ \rho(X_{\overline{Q}}) = 2 \]

\[ \text{Aut } X_{\overline{Q}} = \{1\} \]

no elliptic fibration
\( \tilde{C} \sim 0.012 \)

Cubic: \( 0.9((\log B) - 3)^3 + 10 \)

\( \rho(X) = 3 \)

\( \rho(X_{\overline{Q}}) = 3 \)

\( \# \text{Aut } X_{\overline{Q}} = \infty \) (A. Baragar)

No elliptic fibration

\[
f = -7x^3z + 11x^3w + 7x^2y^2 - 11x^2yz - 11x^2yw - x^2z^2 + 16x^2zw - 9x^2w^2 + 10xy^2z + 5xy^2w - 44xyz^2 + 10xyzw - 14xyw^2 - 10xz^3 + 18x^2z^2w - 15xzw^2 + 5xw^3 + 2y^4 + 23y^3z - 7y^3w + 16y^2z^2 - 3y^2zw + 12y^2w^2 - y^3z^2 - 15yz^2w - 21yzw^2 - 3yw^3 + 18z^4 + 16z^3w + 3z^2w^2
\]
\[ \log(\# \text{ pt's}) \]

\[ C \sim 0.012 \]

line: \[ 3(\log \log B) - 1.2 \]

\[ \rho(X) = 3 \]

\[ \rho(X_{\mathbb{Q}}) = 3 \]

\[ \# \text{Aut } X_{\mathbb{Q}} = \infty \]

no elliptic fibration
$\tilde{C} \sim 0.00061$

parabola: $0.074 \log B^3$

$\rho(X) = 3$

$\rho(X_{\mathbb{Q}}) = 3$

$\# \text{Aut } X_{\mathbb{Q}} = \infty$ (A. Baragar)

no elliptic fibration

\[ f = \cdots \]
\( C \sim 0.00061 \)

line: \( 3(\log \log B) - 2.6 \)

\( \rho(X) = 3 \)

\( \rho(X_{\mathbb{Q}}) = 3 \)

\( \# \text{Aut } X_{\mathbb{Q}} = \infty \)

no elliptic fibration