Cubic points on cubic curves and the Brauer-Manin obstruction on K3 surfaces

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Two problems:

(1) Are there cubic curves without cubic points?

(2) Is the Brauer-Manin obstruction the only one on K3 surfaces?

Goal:

(a) Explain the problems

(b) Relate them
Hasse Principle

Let $X$ be a variety over $\mathbb{Q}$.
If $X$ has no points over $\mathbb{R}$ then $X$ has no points over $\mathbb{Q}$.
If $X$ has no points over $\mathbb{Q}_p$ then $X$ has no points over $\mathbb{Q}$.

Conics satisfy the Hasse principle:
If a conic $C$ has a point over $\mathbb{R}$ and over $\mathbb{Q}_p$ for every $p$, then $C$ has a point over $\mathbb{Q}$.

If a variety $X$ over a number field $k$ has points over every completion of $k$, then we say that $X$ is locally solvable everywhere (LSE).
Cubic curves in general do not satisfy the Hasse principle.

The curve $C$ given by $3x^3 + 4y^3 + 5z^3 = 0$ in $\mathbb{P}^2$ is LSE, but has no points over $\mathbb{Q}$ (Selmer).
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**Question 3**: Does every cubic curve that is LSE have cubic points? (unknown)
Brauer-Manin obstruction.
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$$\mathbb{A}_K = \prod_{v \in M_K} K_v$$

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$$\mathbb{A}_K = \prod_{v \in M_K}^\prime K_v$$

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Let $X$ be a smooth, absolutely irreducible, projective variety over $K$.

Then the set of adèlic points is

$$X(\mathbb{A}_K) = \prod_{v \in M_K} X(K_v)$$

and this is nonempty if and only if $X$ is LSE.
Brauer-Manin obstruction.

For any scheme $Z$ we set $\text{Br } Z = H^2_{\text{ét}}(Z, \mathbb{G}_m)$.
For any ring $R$ we set $\text{Br } R = \text{Br } \text{Spec } R$. 
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For any $K$-algebra $S$ and any $S$-point $x: \text{Spec } S \rightarrow X$, we get a homomorphism $x^*: \text{Br } X \rightarrow \text{Br } S$, yielding a map

$$\rho_S: X(S) \rightarrow \text{Hom}(\text{Br } X, \text{Br } S).$$
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We will apply this to $K$ and to the ring of adèles $\mathbb{A}_K$. 
From class field theory we have

\[ 0 \to \text{Br } K \to \text{Br } \mathbb{A}_K \to \mathbb{Q}/\mathbb{Z} \]

Applying \( \text{Hom}(\text{Br } X, \_ \, ) \) we find \ldots
$0 \to \text{Hom}(\text{Br } X, \text{Br } K) \to \text{Hom}(\text{Br } X, \text{Br } \mathbb{A}_K) \to \text{Hom}(\text{Br } X, \mathbb{Q}/\mathbb{Z})$
$0 \rightarrow \text{Hom}(\text{Br } X, \text{Br } K) \rightarrow \text{Hom}(\text{Br } X, \text{Br } \mathbb{A}_K) \rightarrow \text{Hom}(\text{Br } X, \mathbb{Q}/\mathbb{Z})$
\[0 \to \text{Hom}(\text{Br } X, \text{Br } K) \to \text{Hom}(\text{Br } X, \text{Br } \mathbb{A}_K) \to \text{Hom}(\text{Br } X, \mathbb{Q}/\mathbb{Z})\]
\[ X(\mathbb{A}_K)^{Br} = \psi^{-1}(0) \]

\[ 0 \to \text{Hom}(\text{Br} X, \text{Br} K) \to \text{Hom}(\text{Br} X, \text{Br} \mathbb{A}_K) \to \text{Hom}(\text{Br} X, \mathbb{Q}/\mathbb{Z}) \]
$X(\mathbb{A}_K)^{Br} = \emptyset \implies X(K) = \emptyset$

$0 \to \text{Hom}(\text{Br } X, \text{Br } K) \to \text{Hom}(\text{Br } X, \text{Br } \mathbb{A}_K) \to \text{Hom}(\text{Br } X, \mathbb{Q}/\mathbb{Z})$
\[\begin{align*}
X(\mathbb{A}_K)^{\text{Br}_1} &= \emptyset \quad \Rightarrow \quad X(K) = \emptyset \\
X(\mathbb{A}_K)^{\text{Br}_1} &= \psi_1^{-1}(0) \\
0 &\rightarrow \text{Hom}(\text{Br}_1X, \text{Br } K) \rightarrow \text{Hom}(\text{Br}_1X, \text{Br } \mathbb{A}_K) \rightarrow \text{Hom}(\text{Br}_1X, \mathbb{Q}/\mathbb{Z})
\end{align*}\]

\[\text{Br}_1 X = \ker(\text{Br } X \rightarrow \text{Br } \overline{X})\]
\(X(\mathbb{A}_K)^{\text{Br}(1)} = \emptyset \quad \Rightarrow \quad X(K) = \emptyset.\)

There is a **Brauer-Manin obstruction** to the Hasse principle if

\[X(\mathbb{A}_K) \neq \emptyset \quad \text{and} \quad X(\mathbb{A}_K)^{\text{Br}} = \emptyset.\]
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For a class \( S \) of varieties over \( K \) the Brauer-Manin obstruction is the **only obstruction** to the Hasse principle if for every \( X \in S \) we have

\[ X(\mathbb{A}_K)^{\text{Br}} = \emptyset \quad \Leftrightarrow \quad X(K) = \emptyset. \]
$X(\mathbb{A}_K)^{\text{Br}(1)} = \emptyset \quad \Rightarrow \quad X(K) = \emptyset.$

There is a **Brauer-Manin obstruction** to the Hasse principle if

$$X(\mathbb{A}_K) \neq \emptyset \quad \text{and} \quad X(\mathbb{A}_K)^{\text{Br}} = \emptyset.$$

For a class $S$ of varieties over $K$ the Brauer-Manin obstruction is the **only obstruction** to the Hasse principle if for every $X \in S$ we have

$$X(\mathbb{A}_K)^{\text{Br}} = \emptyset \iff X(K) = \emptyset.$$

**Conjecture**: The Brauer-Manin obstruction is the only obstruction to the Hasse principle for **rationally connected varieties**.
**Definition:** A **K3 surface** is a smooth, absolutely irreducible, projective surface $X$ with trivial canonical sheaf and $H^1(X, \mathcal{O}_X) = 0$.

**Examples of K3’s:**
smooth surfaces of degree 4 in $\mathbb{P}^3$, Kummer surfaces.

**Question 4:** Is the Brauer-Manin obstruction the only obstruction to the Hasse principle for **K3 surfaces**? *(unknown)*
Relating the two problems
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Let $C$ be a smooth cubic curve over $K$ in $\mathbb{P}^2$ and $\rho$ the automorphism

$$\rho: C \times C \to C \times C, \quad (P, Q) \mapsto (Q, R),$$

with $R$ the third intersection point of $C$ with the line through $P$ and $Q$. 
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Let $C$ be a smooth cubic curve over $K$ in $\mathbb{P}^2$ and $\rho$ the automorphism

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with $R$ the third intersection point of $C$ with the line through $P$ and $Q$.

Let $X_C$ be the minimal desingularization of the quotient $(C \times C)/\rho$.

Then $X_C$ is a K3 surface.
**Theorem** (vL)
Let $C$ be the cubic curve in $\mathbb{P}^2_K$ given by $ax^3 + by^3 + cz^3 = 0$ and suppose
(i) $C$ is LSE,
(ii) $abc \in K^*$ is not a cube,
(iii) $C$ has no cubic points (with $K$ as ground field).
Then
$$X_C(\mathbb{A}_K)^{Br_1} \neq \emptyset \quad \text{and} \quad X_C(K) = \emptyset$$
(algebraic Brauer-Manin obstruction is not the only one).
Theorem (vL)
Let $C$ be the cubic curve in $\mathbb{P}^2_K$ given by $ax^3 + by^3 + cz^3 = 0$ and suppose
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(iii) $C$ has no cubic points (with $K$ as ground field).
Then
$X_C(\mathbb{A}_K)^{Br_1} \neq \emptyset$ and $X_C(K) = \emptyset$

sketch of proof:
(iii) implies $X_C(K) = \emptyset$.
Indeed, $T \in X_C(K)$ corresponds to a galois-invariant orbit
$\{(P, Q), (Q, R), (R, P)\}$ of $\rho$ on $C \times C$, so galois acts by even permutations
only and $P, Q, R$ are defined over some cubic extension that is galois.
**Theorem** (vL)

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(i) $C$ is **LSE**, 
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Then

$$X_C(\mathbb{A}_K)^{Br} \neq \emptyset \quad \text{and} \quad X_C(K) = \emptyset$$

**sketch of proof:**

(iii) implies $X_C(K) = \emptyset$.

(ii) implies $Br_1 X_C = Br K$.

Indeed, $Br_1 X_C / Br K \cong H^1(K, \text{Pic } X_C)$, and $\text{Pic } X_C$ is defined over $K(\zeta_3, \sqrt[3]{a/c}, \sqrt[3]{b/c})$, with galois group contained in $(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$. The only subgroups with nontrivial $H^1(K, \text{Pic } X_C)$ all fix $\sqrt[3]{abc}$. 

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Then

$$X_C(\mathbb{A}_K)^{Br_1} \neq \emptyset \quad \text{and} \quad X_C(K) = \emptyset$$

Sketch of proof:

(iii) implies $X_C(K) = \emptyset$.
(ii) implies $Br_1 X_C = Br K$.
(i) implies that $X_C$ is LSE, so $X_C(\mathbb{A}_K) \neq \emptyset$. 
Theorem (vL)
Let $C$ be the cubic curve in $\mathbb{P}_K^2$ given by $ax^3 + by^3 + cz^3 = 0$ and suppose

(i) $C$ is LSE,
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Then

$$X_C(\mathbb{A}_K)^{\text{Br}_1} \neq \emptyset \quad \text{and} \quad X_C(K) = \emptyset$$

**sketch of proof:**

(iii) implies $X_C(K) = \emptyset$.

(ii) implies $\text{Br}_1 X_C = \text{Br} K$.

(i) implies that $X_C$ is LSE, so $X_C(\mathbb{A}_K) \neq \emptyset$.

Done!