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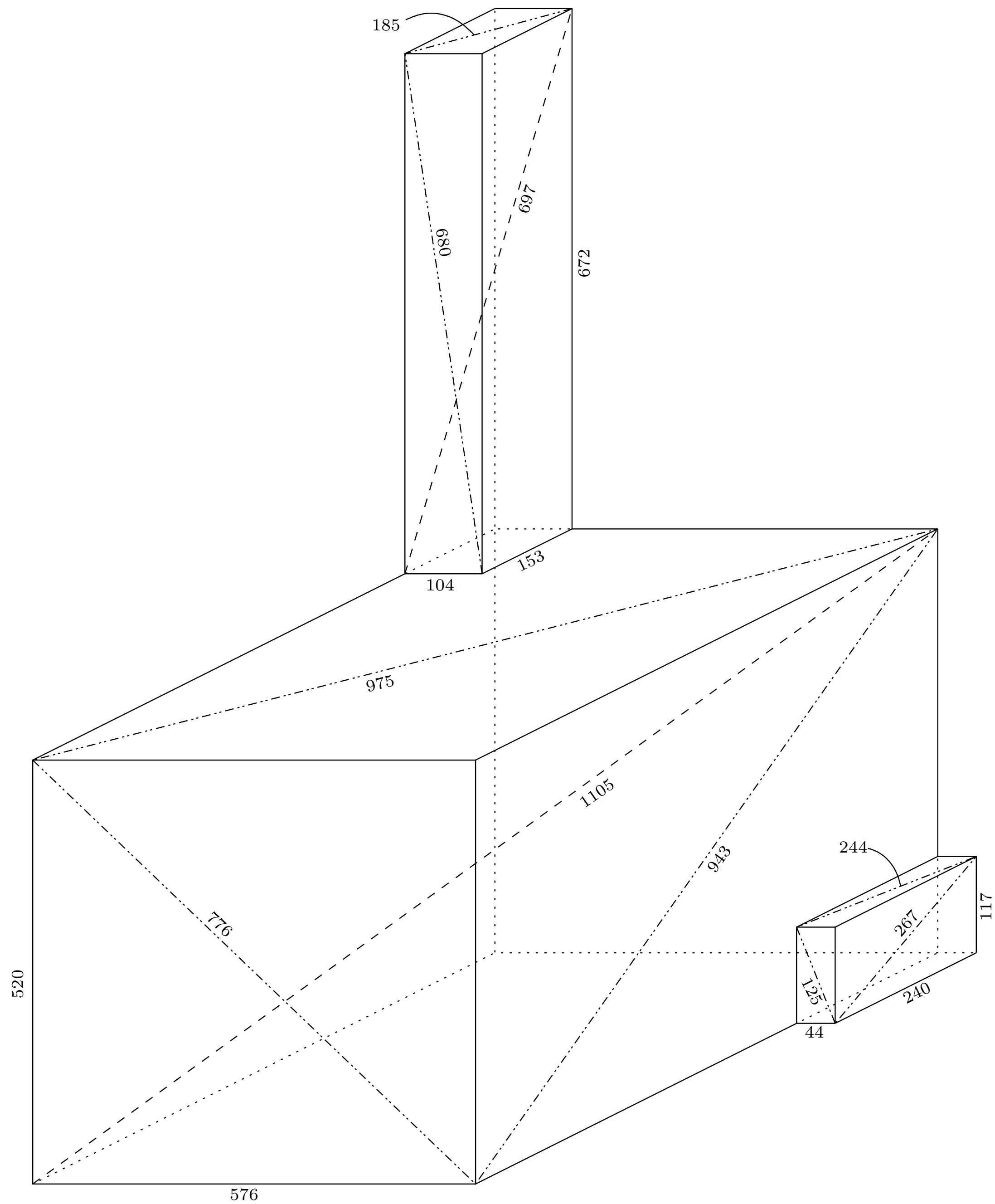

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Contents

1 Introduction

In this paper we will discuss a problem that was already known to Euler and has not been solved ever since. If of a rectangular parallelepiped not only all the sides are integral, but also all the face diagonals, we will call this box a rational cuboid. Euler already knew the smallest rational cuboid which has sides 44, 117 and 240. He also knew that if x, y and z are the sides of a rational cuboid, then so are yz, xz and xy. Applying this trick twice, we get a multiple of the first cuboid. Such cuboids are said to be derived from each other.

A rational cuboid of which also the body diagonal is an integer will be called a perfect cuboid. Although it is hard to believe that Euler never asked himself the question whether perfect cuboids exist, he doesn't seem to have written anything about it. To answer this question has appeared to be a difficult problem. No perfect cuboid has been found so far and a proof of nonexistence doesn't seem to be in sight yet.

This paper will serve two goals. The first is to summarize the literature written about this problem, which is done in section 2. Many papers have been written about (almost) perfect cuboids, only two of which, by F. Beukers and B. van Geemen [1] and by A. Bremner [2], have a more geometrical approach. Therefore the second goal is to analyse the geometrical structure of this problem.

In section 3 we will define a surface Υ of which the rational points correspond with perfect cuboids. We will prove that Υ is a so-called surface of general type, which conjecturally implies that its rational points do not lie Zariski dense. The surface Υ admits many automorphisms, one of which is changing the sign of one of the face diagonals. In section 4 we will divide out by this automorphism to get a K3 surface V. We will compute the Néron-Severi group of a nonsingular model of V. This may help us in finding rational curves on V, thus providing many rational points on V of which we can check whether they lift to rational points on Υ.

2 Literature

As was mentioned in the introduction the problem of finding rational cuboids dates from before Euler. Around 1740 Saunderson [27] already knew that if a, b and c are integers satisfying $a^2 + b^2 = c^2$, then

$$x = 4abc, \qquad y = a(4b^2 - c^2) \qquad \text{and} \qquad z = b(4a^2 - c^2) \tag{1}$$

are the sides of a rational cuboid. Unfortunately for Saunderson cuboids of this form are nowadays usually known as Euler cuboids. Since pythagorean triples are parametrizable by $a = m^2 - n^2$, $b = 2mn$ and $c = m^2 + n^2$, formula (1) gives rise to a 2-dimensional degree 6 parametrization of rational cuboids. One could hope that some of these cuboids are perfect, but Spohn [31] proved that no Euler cuboid can be a perfect cuboid. Spohn [32] was unable to prove completely that a derived cuboid (see introduction) of an Euler cuboid could not be perfect either, but Chein [4] and Lagrange [16] both showed that this could indeed never happen. Although proofs were now already given, Leech [21] gave a short one page proof that no Euler cuboid nor its derived cuboid can be perfect. Spohn [31] proves a theorem that essentially states the easy fact that the (projective) surface given by $x^2 + z^2 = u^2$ and $y^2 + z^2 = v^2$ is parametrized by (1) without the assumption that $a^2 + b^2 = c^2$.

Many people have found almost perfect cuboids, i.e., cuboids of which all but one of the 7 lengths (3 edges, 3 face diagonals and 1 body diagonal) are integral.

Definition 2.0.1 *We will call a cuboid of which all of the seven lengths are integral except possibly for one edge or face diagonal an* edge cuboid *or* face cuboid *respectively. In this context we will also call a rational cuboid a* body cuboid*.*

In section 4 we will take a closer look at face cuboids. Starting with one parametrization, using elliptic curves Colman [5] finds infinitely many two-parameter parametrizations of rational cuboids with rapidly increasing degree. He shows how the same can be done for edge cuboids. He also gives a system of two elliptic curves fibered over a conic of which a rational point would correspond with a perfect cuboid, see section 5.

By elementary examination of the equations for a rational cuboid modulo some small primes, Kraitchik [12] shows that at least one of the sides of a rational cuboid is divisible by 4 and another by 16. Furthermore, the sides are divisible by different powers of 3 and both the primes 5 and 11 divide at least one of the sides. This is equally elementary extended by Horst Bergmann and later by Leech who shows that the product of all the sides and diagonals (edge and face) of a perfect cuboid is divisible by $2^8 \times 3^4 \times 5^3 \times 7 \times 11 \times 13 \times 17 \times 19 \times 29 \times 37$, see Guy's unsolved problems in number theory [8, Problem D18]. Kraitchik [12] also rediscovers the Euler cuboids of (1) and gives a list of 50 rational cuboids that are not Euler cuboids, which he found by some ad hoc methods. He extends his classical list to 241 cuboids with odd side less than 10^6 in [13] and gives 18 more in [14], of which 16 are new.

This was the beginning of an intensive search for rational and in particular for perfect cuboids. Lal and Blundon [17] noted that for rational integers m, n, p and q, the cuboid with sides $x = |2mnpq|$, $y = |mn(p^2 - q^2)|$ and $z = |pq(m^2 - n^2)|$ has at least two integral face diagonals and is rational if and only if $y^2 + z^2 = \square$. Making use of symmetries they had a computer search through all the quadruples (m, n, p, q) with $1 \le m, n, p, q \le 70$ to check if $y^2 + z^2 = \square$. They hereby found 130 rational cuboids, none of which perfect. Later Shanks [28] publishes some corrigenda from their paper.

In [9] I. Korec proves that there are no perfect cuboids with the least side smaller than 10000 in the following way. If x, y and z are the sides of a perfect cuboid, then there are positive integers

a, b and c all dividing x such that with $t = \sqrt{y^2+z^2}$ we have

$$y = \frac{1}{2}\left(\frac{x^2}{a} - a\right), \qquad z = \frac{1}{2}\left(\frac{x^2}{b} - b\right), \qquad t = \frac{1}{2}\left(\frac{x^2}{c} - c\right). \tag{2}$$

Then $c < a, b$ and we can assume $c < b < a < x$. Substituting (2) in $y^2 + z^2 = t^2$, we obtain the equation

$$(a^2c^2 + b^2c^2 - a^2b^2)x^4 - 2a^2b^2c^2x^2 + a^2b^2c^2(a^2 + b^2 - c^2) = 0. \tag{3}$$

From this equation, Korec proves that there are positive integers d and v such that

$$\begin{aligned} d^2 &= (a^2 + b^2)(a^2 - c^2)(b^2 - c^2), \\ v^2 &= abc(abc - d)(a^2 + b^2 - c^2). \end{aligned} \tag{4}$$

Using (4) Korec shows that there are no perfect cuboids for which $c \le 3200$. If we assign weight 1 to a, b and c and weight 3 and 4 to d and v, respectively, then (4) is weighted homogeneous, so in order to prove that (4) implies $c > 3200$, we may assume that $\gcd(a, b, c) = 1$. Then he proves the inequalities

$$a < \frac{bc}{\sqrt{b^2 - c^2}}, \qquad b < c\sqrt{2} \qquad \text{and} \qquad x > (\sqrt{2} + \sqrt{3})c.$$

He also proves that if $p \equiv 3 (\bmod 4)$ is a prime such that $\mathrm{ord}_p(c)$ is odd, then $p^2 < c$. Based on this fact he can exclude many possible values for c. For the remaining values $c \le 3200$ he has a computer run through all values $c < b < c\sqrt{2}$. For each b he checks other conditions that the pair (c, b) has to satisfy, again involving ord_p. For all b for which the pair (c, b) can not be excluded, he has a computer check some necessary p-adic conditions for each a with $b < a < bc/\sqrt{b^2 - c^2}$. It turns out that no 3-tuple (c, b, a) with $c \le 3200$ satisfies all conditions, so for every perfect cuboid we have $c > 3200$, whence $x > (\sqrt{2} + \sqrt{3})c > (\sqrt{2} + \sqrt{3}) \times 3200 > 10000$.

In a later paper Korec [10] proves that there is no perfect cuboid with the least edge smaller than 10^6, using the same ideas as before [9] and the fact that $c > 3200$. For every $x \le 10^6$ he has a computer generate all possible c and b. For each pair (c, b) of the generated possibilities he computes a from (3). If for all the pairs (c, b) the computed a is not an integer dividing x, then x is not an edge of a perfect cuboid. This appeared to be the case for all $x \le 10^6$. To speed up the algorithm he also proves that integers $x \equiv 2 (\bmod 4)$ don't need to be considered as they cannot be an edge of a perfect cuboid. Similarly for x that are prime powers and x of the form $p^m q^n$ with $p \ne q$ prime and either $p \equiv q \equiv 3 (\bmod 4)$ or $m = 1$ or $n = 1$ or $m + n \le 4$. Neither do we need to consider x of the form $2^m q^n$ with q prime and either $q \equiv 3 (\bmod 4)$ or $m \le 3$ or $n = 1$. In order to eliminate as many values for b and c as possible, he proves some theorems that relate $\mathrm{ord}_p(b)$ and $\mathrm{ord}_p(c)$ to $\mathrm{ord}_p(x)$ for primes p and the inequalities $a < x(\sqrt{2}-1)$ and $c < x(\sqrt{3}-\sqrt{2})$ assuming that x is the smallest edge of a perfect cuboid. Unfortunately, Korec makes a mistake in the formula that expresses a in x, c and b and of course it is impossible to check if he made the same mistake in the program that he ran on a computer.

In a third paper Korec [11] computes more lower bounds, this time not based on the smallest edge of a perfect cuboid, but on the full diagonal and therefore also on the largest edge. First he writes the full diagonal z as $z = nq$ where q is a prime. He proves that if z is the full diagonal of a perfect cuboid, then z can only contain primes $1 (\bmod 4)$ and n must be composite. He also proves that for $i = 1, 2, 3$ there exist positive integers a_i, b_i such that $n^2 = a_i^2 + (4b_i)^2$ and that the sides of the perfect cuboid can each be expressed in the a_i and the b_i in one of three different ways. Korec proves that $n \ge 11 \cdot 10^6$ by having a computer run through all the integers $5 \le n \le 11 \cdot 10^6$ with only prime divisors $1 (\bmod 4)$ and checking all the possibilities for the (a_i, b_i). Several conditions for the a_i and the b_i are proven. Some of these are conditions that must hold modulo every positive integer m, some of them are there to avoid double checking, some to rule out some exceptional cases. By a computer search these conditions appeared to be sufficient to conclude that $n > 11 \cdot 10^6$.

This implies that for the interval $I = [z_0, z_1]$ an integer $z \in I$ is not the full diagonal of a perfect cuboid if z contains a prime q with $q > z_1 \cdot (11 \cdot 10^6)^{-1}$ or $q \equiv 3 (\mathrm{mod}\, 4)$. For the remaining $z \in I$ all b_i are contructed such that $z^2 - 16b_i^2 = \square$. The b_i are candidates for the quarters of the even edges and even face diagonals, whence if z is the full diagonal of a perfect cuboid, then there are b_i, b_j and b_k among the candidates such that $b_i^2 + b_j^2 = b_k^2$. This can be checked faster than the straightforward way by partitioning the b_i by the residue class of their squares. Korec let a computer run through several intervals, the union of which covers all positive integers $\leq 8 \cdot 10^9$. No perfect cuboid was found and hence the full diagonal of a perfect cuboid is at least $8 \cdot 10^9$. If x is the maximal edge of a perfect cuboid and z is the full diagonal, then we have $z \leq x\sqrt{3}$, whence $x > \frac{1}{2}z > 4 \cdot 10^9$.

Similar to what Korec [9, 10] does, Rathbun [25, 26] notes that if x, y and z are the sides of a rational cuboid, then $x^2 + y^2 = p^2$ and $x^2 + z^2 = q^2$ for some integers p and q. then $x^2 = (p + y)(p - y) = (q + z)(q - z)$, whence knowing the factorization of x it is possible to recover all possibilities for y and z and thereby all rational cuboids with one side equal to x. Similarly, one can easily find all edge and face cuboids with one side equal to x. Rathbun [25] had a computer check all $x \leq 333,750,000$, which resulted in 6800 body, 6749 face, 6380 edge and no perfect cuboids. He mentions [26] that 4839 of the 6800 body or rational cuboids have an odd side less than $333,750,000$, thereby extending and correcting Kraitchik's classical table [12, 13, 14]. Rathbun claims anno 1999 to have checked all $x \leq 1,281,000,000$ and has not found any perfect cuboid. Leech [22] already noted some errata in Kraitchik's table earlier and Rathbun agrees with this on all but one entry.

J. Leech [18] analyses the problem of existence of all types of cuboids (he notes some minor errata in [19]).

For rational or body cuboids he notes that for the equation $x_i^2 + x_{i+1}^2 = y_{i+2}^2$ there are positive integers a_i, b_i with $\gcd(a_i, b_i) = 1$ such that $x_i : x_{i+1} : y_{i+2} = a_i^2 - b_i^2 : 2a_ib_i : a_i^2 + b_i^2$. Rearranging the x_i if necessary such that x_1, x_2 and x_3 have ascending number of factors 2, we find that exactly one number in both pairs (a_1, b_1) and (a_2, b_2) is even. The numbers a_3 and b_3 are both odd and putting $\alpha = \frac{1}{2}(a_3 + b_3)$, $\beta = \frac{1}{2}(a_3 - b_3)$ we find $x_3 : x_1 = 2\alpha\beta : \alpha^2 - \beta^2$. From $\frac{x_1}{x_2} \cdot \frac{x_2}{x_3} = \frac{x_1}{x_3}$ we find

$$\frac{a_1^2 - b_1^2}{2a_1b_1} \cdot \frac{a_2^2 - b_2^2}{2a_2b_2} = \frac{\alpha^2 - \beta^2}{2\alpha\beta} \tag{5}$$

of which solutions correspond with rational cuboids. Note that we can take the a_i, b_i, α, β to be positive integers such that the generating pairs (a_i, b_i) and (α, β) consist of numbers of opposite parity. Note also that Lal and Blundon [17] listed all cuboids corresponding to solutions of (5) with $a_1, b_1, \alpha, \beta \leq 70$. Leech himself [20] gives all solutions in which two of the pairs a_1, b_1, a_2, b_2, α, β do not exceed 376. Leech [18] also derives the formulas (1) for the Euler cuboids and for the derived cuboid of a rational cuboid. He states that using an infinite descent argument it may be shown that $a_i/b_i = 2$ is impossible for a solution of (5), whence there are no rational cuboids with two edges in the ratio $3 : 4$.

The problem of finding edge cuboids was posed by "Mahatma" [24] and Bromhead [3] extends the cuboid found by readers with sides 124, 957 and $\sqrt{13852800}$ to a one-parameter family of solutions. The smallest real edge cuboid, i.e., with positive non-integral side, has sides 520, 576, $\sqrt{618849}$ and body diagonal 1105. For edge cuboids we are looking for integral solutions of the equations

$$x_1^2 + x_2^2 = y_3^2, \quad t + x_1^2 = y_2^2, \quad t + x_2^2 = y_1^2 \quad \text{and} \quad t + y_3^2 = z^2.$$

Leech notes that for any solution $\xi^2 + \eta^2 = \zeta^2$ one can find τ such that $\tau + \xi^2$, $\tau + \eta^2$ and $\tau + \zeta^2$ are all squares by Fermat's method's for "triple equations" (see [48], p. 321–328), giving the solution

$$\tau = (\zeta^8 - 6\xi^2\eta^2\zeta^4 + \xi^4\eta^4)/(2\xi\eta\zeta)^2. \tag{6}$$

This was also found by D. Zagier and J. Top. Leech also shows that then $(\zeta^4 - \xi^2\eta^2)^4 - y_3^4 = \tau(\zeta^4 + \xi^2\eta^2)^2$ and as the difference of two fourth powers cannot be a square, τ cannot be either. One could also look at the equations

$$x_1^2 + x_2^2 = y_3^2, \quad z^2 = x_1^2 + y_1^2 = x_2^2 + y_2^2.$$

Fixing z we are looking for x_1 and x_2, both appearing in some representation $z^2 = x^2 + y^2$ (we can find all these representations from the factorization of z) such that $x_1^2 + x_2^2$ is a square not equal to z^2 (equality would yield $t = 0$). Leech [20] finds by a computer search over all composite z with no prime divisors $p \equiv 3 (\mathrm{mod}\, 4)$ that there are 160 solutions with $z \leq 250000$ of which 78 have $t > 0$. Similar to the situation of rational cuboids, there are a_i, b_i, α and β such that

$$\frac{x_2}{x_1} = \frac{a^2 - b^2}{2ab}, \quad \frac{z}{x_1} = \frac{\alpha_1^2 - \beta_1^2}{2\alpha_1\beta_1}, \quad \frac{z}{x_2} = \frac{\alpha_2^2 - \beta_2^2}{2\alpha_2\beta_2},$$

of course satisfying the equation given by $\frac{x_2}{z} \cdot \frac{z}{x_1} = \frac{x_2}{x_1}$. Putting $u = \alpha_1/\beta_1$ and $v = \alpha_2/\beta_2$ we find for fixed $x_2/x_1 = (a^2 - b^2)/(2ab)$ the elliptic curve

$$E_{x_2/x_1} \colon x_1 v(u^2 + 1) = x_2 u(v^2 + 1) \tag{7}$$

on which the point $P = (u_0, v_0) = (a/b, (a + b)/(a - b))$ has infinite order. Leech says that the trivial solution $t = 0$ corresponds to P, but in fact by taking inverses of u_0 or v_0 one obtains three other points that correspond with $t = 0$. They differ from P by a torsionpoint. This gives infinitely many rational solutions with the same ratio x_2/x_1. The point $2P$ corresponds to the solution (6). Leech remarks that many of the solutions with $z \leq 10^5$ occur in cycles of four, i.e., for $i = 1, 2, 3, 4$ there are ξ_i and η_i such that $z^2 = \xi_i^2 + \eta_i^2$ and $\xi_i^2 + \xi_{i+1}^2 = \Box$. He also notes (again empirically) that $\xi_1\xi_3 = \xi_2\xi_4$ for those cycles, so $\xi_2/\xi_1 = \xi_3/\xi_4$, whence the two solutions we get from $\xi_1^2 + \xi_2^2 = \Box$ and $\xi_3^2 + \xi_4^2 = \Box$ correspond to points on the same elliptic curve E as in (7). For all cycles of four Leech notes that the line through these two points goes through the trivial point corresponding to $t = 0$. He states that by brute force the converse may be proven, namely that for two points on the same elliptic curve on such a line we get such a cycle of four and each non-trivial solution belongs to such a cycle. For instance, the point $Q = (u, v) = (\frac{a}{b}, \frac{a-b}{a+b})$, one of the points corresponding to $t = 0$ on the elliptic curve (7) for $x_2/x_1 = p/q$ with $p = a^2 - b^2$, $q = 2ab$ and $r = a^2 + b^2$, and the point $2Q$, corresponding to (6), form a cycle of four. We get $2Q = (\frac{q^2}{pr}, \frac{p^2}{qr})$, also corresponding to a cuboid with $x_2/x_1 = p/q$. Of the other two cuboids in the cycle there is one with $\frac{\alpha_1}{\beta_1} = \frac{a}{b}$ and $\frac{\alpha_2}{\beta_2} = \frac{q^2}{pr}$ as the generators of $\xi_i^2 + \eta_i^2 = \zeta^2$, i.e., $\xi_i : \eta_i = 2\alpha_i\beta_i : \alpha_i^2 - \beta_i^2$. The other has $\frac{\alpha_1}{\beta_1} = \frac{a-b}{a+b}$ and $\frac{\alpha_2}{\beta_2} = \frac{p^2}{qr}$ as the generators of $\xi_i^2 + \eta_i^2 = \zeta^2$. These both have $x_1 : y_1 : z = q : p : r$ so we conclude that every pythagorean triangle can occur as a body diagonal, an edge and a face diagonal of some edge cuboid. From the cuboid corresponding to $2Q$ we can conclude that every pythagorean triangle occurs as the diagonal and two edges of some edge cuboid. Mind that that Leech [18] sometimes mixes up $\frac{\beta_i}{\alpha_i}$ with $\frac{\alpha_i}{\beta_i}$. On some elliptic curves (7) there are more points than generated by torsionpoints and points corresponding to $t = 0$. The two extra cuboids in the cycle just described correspond to points on the elliptic curve for which $x_1^2 + x_2^2 = y_3^2$ is generated by a and b with $\frac{a}{b} = \frac{pq}{r^2}$. Leech gives a few more, contained in the cycle for which the generators for $\xi_1^2 + \xi_2^2 + \zeta^2$ are

$$\frac{\alpha_i}{\beta_i} = \frac{mn}{m^2 - n^2}, \quad \frac{m^3 - n^3}{m^3 + n^3}, \quad \frac{n(m^4 + m^2n^2 + n^4)}{m(m^4 - m^2n^2 - n^4)}, \quad \frac{2m^2n^2}{m^4 - m^2n^2 - n^4},$$

and those for the ratios $\xi_i/\xi_{i\pm1}$ are

$$\frac{b}{a} = \frac{mn}{m^2 + n^2}, \quad \frac{n(m^4 - m^2n^2 - n^4)}{m(m^4 - m^2n^2 + n^4)}.$$

Leech [18] mentions that the problem of the face cuboids is related with the problem of finding three integers all pairs of which have have their sums and differences squares, which is equivalent to finding three squares whose differences are squares, see Dickson [6], ch. 15, ref. 28 and ch. 19, refs. 40–45. For a face cuboid, i.e., a solution of

$$x_1^2 + x_2^2 = y_3^2, \qquad x_3^2 + x_1^2 = y_2^2, \qquad x_2^2 + y_2^2 = z^2, \tag{8}$$

the integers $2(z^2+x_1^2)$, $2(z^2-x_1^2)$ and $2|x_2^2-x_3^2|$ satisfy the first problem, the squares of z, y_3 and x_2 the second. Leech doesn't mention that the problems are actually equivalent with finding face cuboids, any solution of the second problem gives squares we can take to be the squares of z, y_3 and x_2. Similar to what we have seen before, the generators for the equations (8) are integers α_i, β_i, for $i = 1, 2, 3$ satisfying $\frac{y_2}{x_2} \cdot \frac{x_2}{x_1} = \frac{y_2}{x_1}$, i.e.

$$\frac{\alpha_1^2 - \beta_1^2}{2\alpha_1\beta_1} \cdot \frac{\alpha_3^2 - \beta_3^2}{2\alpha_3\beta_3} = \frac{\alpha_2^2 + \beta_2^2}{2\alpha_2\beta_2}. \tag{9}$$

Writing $u_i = \{(\alpha_i^2 - \beta_i^2)/(2\alpha_i\beta_i)\}^2$, this becomes $u_1u_3 = 1 + u_2$, a recursive relation with period 5, discussed by Lyness [23] in the context of finding three integers whose pairs have sums and differences squares. Exchange of the first equations of (8) and then of the left factors of (9) gives a new solution corresponding to $u_2u_4 = 1 + u_3$, whence solutions come in cycles of five. The u_i in a cycle correspond to the squares of the values y_2/x_2, x_3/x_1, x_2/x_1, y_3/x_3 and x_1z/x_2x_3 of one of the five occurring cuboids. Leech [20] gives all solutions in which two of the pairs α_i, β_i do not exceed 376.

Leech [18] also shows that for a face cuboid there exist integers p, q, r and s such that

$$\begin{aligned} x_1 &= ps - qr, \quad y_2 = ps + qr, \quad y_3 = pr - qs, \quad z = pr + qs, \\ x_2^2 &= (p^2 - q^2)(r^2 - s^2), \quad x_3^2 = 4pqrs, \quad \frac{\beta_5^2}{\alpha_5^2} = \frac{p^2 - q^2}{2pq}\frac{2rs}{r^2 - s^2}. \end{aligned} \tag{10}$$

The numbers $p, q, r, s = 3^2, 2^2, 9^2, 7^2$ give a solution which has the smallest face cuboid with edges 104, 153 and 672 in its cycle of five. Another way of finding solutions is by noting that $z^2 = x_2^2 + y_2^2 = x_3^2 + y_3^2$ can be written as the sum of two squares in two different ways. For each z we can find all the pairs (a_i, b_i) such that $z^2 = a_i^2 + b_i^2$ and check if there are a_i and b_j such that $b_j^2 - a_i^2 = \Box$. This enables Leech [20] to give all 89 face cuboids with $z < 10^5$.

He concludes from (10) that a pair of generators α, β occurs in some face cuboid if and only if β^2/α^2 is expressible as the product or quotient of two ratios of the form $(p^2-q^2)/2pq$. This would give a method of search for generators which can occur in solutions. Combining those would yield all solutions. Strangely enough he says that this is not the case. Not even all cycles would be found because some cycles, such as

$$\frac{\beta}{\alpha} = \frac{4}{13}, \quad \frac{1}{13}, \quad \frac{1}{9}, \quad \frac{14}{27}, \quad \frac{16}{21} \tag{11}$$

would not include pairs of generator pairs with ratios whose squares are expressible as the product and quotient of a pair of ratios of the form $(p^2-q^2)/2pq$ according to Leech. This doesn't seem to be true since for the 4-tuples $(p, q, r, s) = (65, 37, 481, 5)$,$(45, 37, 333, 5)$, $(169, 41, 41, 1)$, $(81, 49, 9, 4)$ and $(81, 17, 17, 1)$ that we get from (10), the quotient of $(p^2 - q^2)/2pq$ and $(r^2 - s^2)/2rs$ are the squares of the values in (11). And if x can be written as the quotient of two ratios of the form $(p^2 - q^2)/2pq$ then it can also be written as the product of such ratios, for we have

$$\frac{p^2 - q^2}{2pq} = \frac{2(p+q)(p-q)}{(p+q)^2 - (p-q)^2}.$$

Using techniques of Diophantus' Arithmetica Leech finds a parametric solution of (8), given by

$$x_1 = 4rst(s+t), \quad x_2 = 2rt^2(t+2s), \quad x_3 = s(s+t)(2r+t)(2r-t). \tag{12}$$

with $r = m^2 + m + 1$, $s = 2m + 1$, $t = m^2 - 1$, which corresponds to the cycle

$$\frac{\beta_i}{\alpha_i} = \frac{r(2s+t)}{t^2}, \quad \frac{t}{2r}, \quad \frac{s+t}{s}, \quad \frac{(t-r)(2r+t)}{(t+r)(2r-t)}, \quad \frac{t(2s+t)}{(2r-t)(2r+t)}. \tag{13}$$

Leech seems to write occurring generating fractions β/α in such a way that $\beta \leq \alpha$. Using a parametric expression for ratios of the form $(p^2 - q^2)/2pq$ whose product and quotient are squares he finds the same cycle.

For fixed β_2 and α_2 substitution of $u = \alpha_1/\beta_1$ and $v = (\alpha_3 + \beta_3)/(\alpha_3 - \beta_3)$ gives the elliptic curve

$$E\colon \quad (\alpha_2^2 + \beta_2^2)u(v^2 - 1) = 2\alpha_2\beta_2 v(u^2 - 1). \tag{14}$$

Using a descent argument we can show again that there are no solutions with $\alpha/\beta = 2$ or 3, i.e., none with edges in the ratio $3 : 4$. A solution of (8) gives a point P on E for α_5, β_5. The point $-2P$ then corresponds to a cycle of solutions including

$$\frac{\beta}{\alpha} = \frac{y_2 - y_3}{y_2 + y_3}, \quad \frac{\beta_5}{\alpha_5}, \quad \frac{x_3}{x_2}, \quad \frac{x_1}{z}.$$

The fifth pair corresponds to

$$\{zx_1(y_2^2 - y_3^2)\}^2 + \{x_2x_3(y_2^2 + y_3^2)\}^2 = \{y_2y_3(x_2^2 + x_3^2)\}^2.$$

Using this together with (13) we find that there are infinitely many rational curves on the surface in $\mathbb{P}^5$ described by the equations in (8).

If a, b occurs as a pair of generators of $x_i^2 + x_{i+1}^2 = y_{i-1}^2$ in a perfect cuboid, then the ratios of any two of $a^2 - b^2, 2ab, a^2 + b^2$ is expressible as the product of two ratios of the form $(p^2 - q^2)/2pq$. Not many pairs a, b seem to satisfy this.

Leech then digresses on spherical triangles and the question of existence of four squares all whose sums or all whose differences in pairs are square. If they exist then certain cycles are contained in a constructed graph containing solutions for three squares in stead of four.

He concludes with a list of open problems. We will phrase all of those that imply (non)-existence of perfect cuboids. The first two are equivalent with the problem of finding perfect cuboids, the others are soluble if a perfect cuboid exists.

- Is there a ratio of the form $(p^2 + q^2)/2pq$ and one of the form $(p^2 - q^2)/2pq$ whose product and quotient are both of the form $(p^2 - q^2)/2pq$?
- Are there non-trivial solutions of $(a^2c^2 - b^2d^2)(a^2d^2 - b^2c^2) = (a^2b^2 - c^2d^2)^2$?
- Are there integers ζ and ξ_i, η_i for $i = 1, 2, 3$ with $\zeta^2 = \xi_2 + \eta^2$ and $\xi_i^2 + \xi_{i+1}^2 = \square$, even without the condition $\xi_1^2 + \xi_2^2 + \xi_3^2 = \zeta^2$?
- Is there a sequence of integers $x_0, x_1, x_2, x_3, \ldots$ such that for any two and three adjacent elements the sum of their squares is a square? We could take $x_1, x_2, x_3, x_1, x_2, x_3, x_1, \ldots$ for a perfect cuboid. A finite sequence of length 8 exists.
- Is there a rational cuboid with sides x_1, x_2, x_3 and an integer z such that $z - x_i^2 = \square$?
- Is there a rational cuboid with sides x_1, x_2, x_3 and an integer z such that $z - x_i^2 - x_{i+1}^2 = \square$?
- Are there four squares all pairs of which have square differences? For a perfect cuboid we could take the squares of y_3z, y_2y_3, x_1z and x_1y_3.
- Is there a sequence of four squares of which the sums of all $n \leq 4$ adjacent elements is a square? Here we could take the squares of x_1y_3, x_1x_3, x_2x_3 and x_2y_3.
- Is there a cycle of solutions of (9) in which $\alpha_1/\beta_1 = (p^2 - q^2)/2pq$ and $\alpha_2/\beta_2 = (r^2 + s^2)/2rs$?

A. Bremner [2] analyses the surface $V \subset \mathbb{P}^5$ given by

$$X^2 + Y^2 = R^2, \quad Y^2 + Z^2 = S^2, \quad Z^2 + X^2 = T^2,$$

describing rational cuboids. He proves that V is birationally equivalent with the surface $W \subset \mathbb{P}^3$ given by

$$(x^2 - y^2)(z^2 - t^2) = 2yz(x^2 - t^2).$$

He gives all lines and conics on the surface W. It turns out that there are precisely 22 straight lines on W, of which 14 are defined over $\mathbb{Q}$ and 8 defined over $\mathbb{Q}(\sqrt{2})$. Furthermore, he finds all parametrizable plane cubics on W. There are infinitely many rational curves on W, of which Bremner gives several. Finally, he gives five degree 8 parametrizations of curves on V of which he claims three appear to be new. The other two were already found by Kraitchik [14]. All five correspond to points on the same elliptic curve over $\mathbb{Q}(\lambda)$.

F. Beukers and B. van Geemen [1] analyse two surfaces V and W describing face and rational cuboids respectively, namely

$$V = \begin{cases} a^2 + b^2 & = x^2 \\ a^2 + c^2 & = y^2 \\ a^2 + b^2 + c^2 & = u^2 \end{cases} \quad \text{and} \quad W = \begin{cases} a^2 + b^2 & = x^2 \\ a^2 + c^2 & = y^2 \\ b^2 + c^2 & = z^2 \end{cases} \tag{15}$$

They find infinitely many rational curves on both surfaces. To analyse the surface V, their first step is to show that it is birationally equivalent with the affine surface in $\mathbb{A}^3$ given by

$$u^2 = (p^4 + q^2)(q^2 + 1).$$

Let E be the projective elliptic curve

$$E\colon y^2 z = x^3 - 4xz^2$$

with torsion point $T = (0:0:1)$. Let $G \cong (\mathbb{Z}/2)^2$ be the group of automorphisms of the abelian surface $E \times E$ generated by $\tau\colon (P,Q) \mapsto (P+T, Q+T)$ and $\iota\colon (P,Q) \mapsto (-P,-Q)$. Then F. Beukers and B. van Geemen show that there is an isomorphism

$$\Phi\colon (E \times E)\,/G \to V.$$

They also show that if C is a (hyper-)elliptic curve and

$$A\colon C \to E \qquad \text{and} \qquad B\colon C \to E$$

are rational maps commuting with the standard involutions on C and E, then the image of the composite map $P \mapsto \Phi(A(P), B(P))$ from C to V is a rational curve. Taking A and B to be endomorphisms of E and other concrete examples they find four degree ≤ 10 rational curves over $\mathbb{Q}$ on V and one degree 6 rational curve over $\mathbb{Q}(i)$. Let E' be the elliptic curve $E/\langle P \mapsto P+T \rangle$. Then E' is given by $E'\colon y^2 z = x^3 + xz^2$ and it is proven that V is the Kummer surface of $E \times E'$, i.e., V is isomorphic over $\mathbb{Q}$ to $E \times E'$ modulo the involution $\mathrm{inv}\colon (P,Q) \mapsto (-P,-Q)$. In section 4 we will further analyse the surface V.

They also show that the surface W is birationally equivalent with the pencil of elliptic curves

$$y^2 = x(x+1)(x+\lambda^2), \qquad \lambda = (\sigma - \sigma^{-1})/2.$$

The Euler cuboids correspond with the points

$$x = \frac{(3t^2-1)^2}{(t^2-1)(t^2+1)^2}, \quad y = \frac{9t^9 + 84t^7 - 122t^5 + 52t^3 - 7t}{4(t^2-1)^2(t^2+1)^3}, \quad \sigma = 2\frac{t^2-1}{t^2+1}.$$

Addition on this elliptic curve gives rise to infinitely many rational curves on W.

3 Classification of surfaces and an application

As was announced, this paper will have a more geometrical approach than most of the articles described in the previous section. In finding perfect cuboids we are actually looking for rational points on a surface that will be described in section 3.2. It is easy to find some rational points on this surface, namely those corresponding to cuboids with zero contents.

Finding rational points on an algebraic surface, or even deciding whether they exist, is in general a difficult problem and before we start with a search one might ask the question of how many points to expect. For a curve C over $\mathbb{Q}$ we know by Faltings' Theorem that if the genus g_C is at least 2, then there are only finitely many rational points. In other words, the rational points do not lie Zariski dense on the curve. If the genus is at most one, then there can be infinitely many rational points on the curve which in that case are Zariski dense.

A generalization for higher dimensional varieties has not been proved yet, but there are certain conjectures that would generalize the situation for curves. We will discuss these in section 3.4. However, for a higher dimensional variety X neither the arithmetic, nor the geometric genus turns out to be the appropriate invariant to consider. Instead we should look at the so-called *Kodaira dimension* of X. The Kodaira dimension $\kappa(X)$ is a number satisfying $-1 \le \kappa(X) \le \dim X$. For a curve C with genus 0, 1 or ≥ 2 we have $\kappa(C) = -1, 0$ or 1 respectively. Hence Faltings' Theorem states that if the Kodaira dimension of a curve C defined over $\mathbb{Q}$ is 1, then the set rational points on C is not Zariski dense.

In this form Faltings' Theorem generalizes directly to the Weak Lang Conjecture which says that if X is a variety over $\mathbb{Q}$ and the Kodaira dimension $\kappa(X)$ equals $n = \dim X$, then the set of rational points on X is not Zariski dense. Varieties satisfying $\kappa(X) = \dim X$ will be called of *general type*.

In section 3.2 we will define a surface Υ describing perfect cuboids and we will state and prove some of its properties. Then in section 3.3 we will define the Kodaira dimension of a variety and we will see that Υ is indeed of general type. In section 3.4 we will try to find out what this means for the rational points on Υ.

We will have to do some preparation before we can define the notion of Kodaira dimension. This will be done in section 3.1. In this section we will mainly state several definitions and propositions from Chapter II of R. Hartshorne's *Algebraic Geometry* [47] and a partial generalization of Exercise II.8.4 of that book. In the same section we will also state some definitions and propositions about Hodge numbers. The Hodge numbers only return in Proposition 3.2.19, Corollary 3.3.34 and Bluff 1 and 2. Therefore the reader who is not interested in these results and who is familiar with Chapter II of [47] can skip section 3.1, apart maybe from Lemma 3.1.36, and continue directly with section 3.2. Note that rings are taken to be commutative with 1.

3.1 Definitions and tools

We will recall some algebraic geometrical notions such as divisors and the relation with invertible sheaves with references for the proofs. We will not deal with these notions in their full generality, but only to the extend needed. To avoid confusion about the notion variety, note that we take the following definitions, following R. Hartshorne [47].

Definition 3.1.1 *A variety over a field K is a geometrically integral separated scheme, which is of finite type over K. Curves and surfaces are varieties of dimension* 1 *and* 2 *respectively.*

Definition 3.1.2 *A* projective variety *over K is a variety which is a closed subscheme of* $\mathbb{P}^n_K$.

Definition 3.1.3 *A variety X over an algebraically closed field K is* nonsingular *if all its local rings are regular local rings.*

The Kodaira dimension of a nonsingular variety is defined in terms of its canonical sheaf, which is an invertible sheaf. Invertible sheaves on X are closely related with divisors and line bundles on X. We will give a short overview.

Let X be a noetherian integral separated scheme which is regular in codimension one. Then we can define the notion Weil divisor, see [47, II.6] for details and proofs.

Definition 3.1.4 *A* prime divisor *on a noetherian integral separated scheme X which is regular in codimension 1 is a closed integral subscheme of codimension 1. A* Weil Divisor *D on X is an element of the free abelian group* $\operatorname{Div} V$ *generated by the prime divisors.*

If for all the coefficients n_Y of a Weil divisor $D = \sum_Y n_Y \cdot Y$ we have $n_Y \geq 0$ then we call D effective, writing, $D \geq 0$. We write $D_1 \geq D_2$ if $D_1 - D_2 \geq 0$. For the generic point η of a prime divisor Y the local ring $\mathcal{O}_{\eta,X}$ is a discrete valuation ring with quotientfield $K(X)$, the function field of X. Let v_Y denote the corresponding valuation. For any nonzero rational function $f \in K(X)^*$ on X there are only finitely many prime divisors Y with $v_Y(f) \neq 0$, so we can define the divisor $(f) = \sum_Y v_Y(f) \cdot Y$. This gives a homomorphism

$$K(X)^* \to \operatorname{Div} X: \quad f \mapsto (f).$$

The divisors in the image of this homomorphism are called *principal* divisors. The group $\operatorname{Div} X$ of all divisors divided by the subgroup of principal divisors is called the *divisor class group* of X, and is denoted by $\operatorname{Cl} X$. Two divisors with the same divisor class, i.e., that differ a principal divisor, are called *linearly equivalent.* If Y is an integral subscheme of X given by the inclusion $j\colon Y \hookrightarrow X$, then we have a homomorphism $j^*\colon \operatorname{Div} X \to \operatorname{Div} Y$ given by $\sum n_Z \cdot Z \mapsto \sum n_Z(Z \cap Y)$, where we leave out those Z with $Z \cap Y = \emptyset$. The morphism j^* maps principal divisors to principal divisors and hence induces a map $j^*\colon \operatorname{Cl} X \to \operatorname{Cl} Y$. We have the following proposition.

Proposition 3.1.5 *Let Y be a noetherian integral separated scheme, which is regular in codimension one. Let Z be a proper closed subset of Y and let $V = Y - Z$. Then:*

(i) there is a surjective homomorphism $\operatorname{Cl} Y \to \operatorname{Cl} V$ defined by $D = \sum n_i D_i \mapsto \sum n_i(D_i \cap V)$, where we ignore those $D_i \cap V$ which are empty;

(ii) if $\operatorname{codim}(Z, Y) \geq 2$, then $\operatorname{Cl} Y \to \operatorname{Cl} V$ is an isomorphism.

(iii) if Z is an irreducible subset of codimension 1, then there is an exact sequence

$$0 \to \mathbb{Z} \to \operatorname{Cl} Y \to \operatorname{Cl} V \to 0,$$

where the first map is defined by $1 \mapsto 1 \cdot Z$.

Proof. See [47, Prop.II.6.5]. □

For arbitrary schemes we can define the notion *Cartier divisor*, but we will only deal with integral schemes here. Hence let X be an integral scheme, $\mathcal{O}_X$ its structure sheaf and $\mathcal{K}_X$ the constant sheaf corresponding to the functionfield $K(X)$ of X, i.e., the local ring $\mathcal{O}_\xi$ of the generic point ξ of X (see [47, Exc.II.3.6]). Let $\mathcal{O}_X^*$ and $\mathcal{K}_X^*$ denote the sheaves of (multiplicative groups) of invertible elements in $\mathcal{O}_X$ and $\mathcal{K}_X$ respectively.

Definition 3.1.6 *We define a* Cartier divisor *on the integral scheme X to be a global section of the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$.*

Remark 3.1.7 To see how Cartier divisors are defined on arbitrary schemes, see [47, II.6]. See also [60, II.4] and [52, I.1].

The multiplicative group structure of the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$ induces a group structure on the set $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ of all Cartier divisors, which we will write additively. The principal divisors are those in the image of the natural map $\Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. The quotient is denoted $\mathrm{CaCl}(X)$.

A Cartier divisor D can be given by $\{(U_i, f_i)\}$ where $\{U_i\}$ is an open cover of X and $f_i \in K(X)^*$, such that $f_i f_j^{-1} \in \mathcal{O}_X^*(U_i \cap U_j)$. If f_i is regular on U_i for all i then we call the Cartier divisor effective and write $D \geq 0$. For two Cartier divisors D_1, D_2 we write $D_1 \geq D_2$ if $D_1 - D_2 \geq 0$. If X is not only integral, but also noetherian, separated and regular in codimension one and $D = \{(U_i, f_i)\}$ is a Cartier divisor, then we can define the associated Weil divisor by taking the coefficient of each prime divisor Y to be $v_Y(f_i)$, where i is any index for which $Y \cap U_i \neq \emptyset$. As X is noetherian, the sum $\sum v_Y(f_i)Y$ is finite and gives us a well-defined Weil divisor D' on X, which is effective if and only if the original Cartier divisor D is effective.

Proposition 3.1.8 *If X is a noetherian integral separated normal scheme, then this induces an injective homomorphism*

$$\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \to \mathrm{Div}\, X$$

from the group of Cartier divisors to the group of Weil divisors, where principal Cartier divisors correspond to principal divisors in the image. Hence it induces an injective homomorphism

$$\mathrm{CaCl}\, X \to \mathrm{Cl}\, X.$$

If X is even locally factorial, then both these homomorphisms are actually isomorphisms. This holds in particular in the case that X is a non-singular projective variety over a field K.

Proof. See [47, II.6.11] and [52, I.1]. □

Definition 3.1.9 *Let X be an integral scheme and let D be a Cartier divisor on X, given by $\{(U_i, f_i)\}$. We define a subsheaf $\mathcal{L}(D)$ of $\mathcal{K}_X$ as the sub-$\mathcal{O}_X$-module of $\mathcal{K}_X$ generated by f_i^{-1} on U_i, i.e., for $U \subset U_i$ we get $\mathcal{L}(D)(U) = f_i^{-1}\mathcal{O}_X(U)$.*

This sheaf is clearly locally isomorphic to $\mathcal{O}_X$, whence an invertible sheaf. This map $D \mapsto \mathcal{L}(D)$ induces an injective homomorphism from $\mathrm{CaCl}\, X$ to $\mathrm{Pic}\, X$, the group of invertible sheaves on X up to isomorphism.

Proposition 3.1.10 *If X is integral, then this homomorphism $\mathrm{CaCl}\, X \to \mathrm{Pic}\, X$ is an isomorphism.*

Proof. See [47, Prop. II.6.15]. □

Corollary 3.1.11 *If X is a noetherian integral separated normal scheme then we get an injective homomorphism $\mathrm{Pic}\, X \to \mathrm{Cl}\, X$, which is an isomorphism if X is also locally factorial, in particular if X is regular.*

Proof. This follows from Propositions 3.1.8 and 3.1.10. □

If D is an effective divisor on a scheme X, then $\mathcal{L}(-D)$ is an invertible sheaf that we are already familiar with.

Lemma 3.1.12 *Let D be an effective Cartier divisor on a scheme X, and let Z be the associated locally principal closed subscheme. Let $\mathcal{I}_Z$ be the corresponding sheaf of ideals on X. Then $\mathcal{I}_Z = \mathcal{L}(-D)$.*

Proof. See [47, Prop.II.6.18]. □

Definition 3.1.13 *A* (geometric) line bundle *L on a variety X over a field K is a variety L with a morphism $f: L \to X$ together with an open covering $\{U_i\}$ of X and isomorphisms $\psi_i: f^{-1}(U_i) \to U_i \times \mathbb{A}^1$ such that for any i, j and any affine $V \subset U_i \cap U_j$ with coordinate ring A we get an isomorphism $\psi = \psi_j \circ \psi_i^{-1}: V \times \mathbb{A}^1 \to V \times \mathbb{A}^1$ that can be given by $\psi: (P, x) \mapsto (P, ax)$ for some $a \in A$, independent from $P \in V$ and $x \in \mathbb{A}^1$. Hence every fibre L_x over a point $x \in X$ is a one-dimensional vectorspace over K.*

To a line bundle L we can associate an invertible sheaf $\mathcal{L}(L)$, called the *sheaf of sections of* L, see [47, Exc.II.5.18], where all this is done for arbitrary schemes. A *section* of $f: L \to X$ over an open set $U \subset X$ is a morphism $s: U \to L$ such that $f \circ s = \mathrm{id}_U$. The presheaf $\mathcal{L}(L): U \mapsto$ $\{$set of sections of f over $U\}$ is a sheaf of sets that has a natural structure of $\mathcal{O}_X$-module, which makes it an invertible sheaf.

Proposition 3.1.14 *The map $L \mapsto \mathcal{L}(L)$ described above induces a one-to-one correspondence between the isomorphism classes of line bundles and* $\operatorname{Pic} X$*, the group of isomorphism classes of invertible sheaves.*

Proof. See [47, Exc.II.5.18]. □

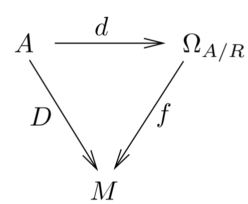
Note that for every Cartier divisor class $[D]$ we now have a corresponding isomorphism class $[L]$ of line bundles and an isomorphism class $[\mathcal{L}]$ of invertible sheaves with $\mathcal{L} \cong \mathcal{L}(D) \cong \mathcal{L}(L)$. If X is a normal variety, then we can even associate a Weil divisor class to $[D]$.

Let X be a non-singular variety of dimension n. Then for every point $P \in X$ the tangent space to X in P is a vector space of dimension n. The n-th exterior power of the dual space gives a one dimensional vectorspace in every point of X. This defines a line bundle, the dual of which we will call the *canonical line bundle*. In terms of sheafs we can define the *canonical sheaf* ω_X as in Definition 3.1.23. Let us first restate some properties of derivations, see [50, Sect.XIX.3].

Definition 3.1.15 *Let R be a ring, A an R-algebra and M an A-module. By a* derivation *$D: A \to M$ (over R) we mean an R-linear map satisfying the usual rules $D(ab) = aD(b) + bD(a)$.*

Remark 3.1.16 Note that $D(1) = D(1 \cdot 1) = 2D(1)$, so $D(1) = 0$, whence $D(R) = 0$. Such derivations form an A-module $\mathrm{Der}_R(A, M)$ in a natural way, where aD is defined by $(aD)(b) = aDb$.

Definition 3.1.17 *By a* universal derivation *for A over R, we mean an A-module $\Omega = \Omega_{A/R}$, and a derivation $d: A \to \Omega$, such that, given a derivation $D: A \to M$, there exists a unique A-homomorphism $f: \Omega \to M$ making the following diagram commutative.*



Remark 3.1.18 By this definition the A-module $\Omega_{A/R}$ can be characterised by

$$\mathrm{Hom}_A\left(\Omega_{A/R}, M\right) \cong \mathrm{Der}_R\left(A, M\right).$$

The following proposition shows that the universal derivation exists.

Proposition 3.1.19 *Let A be an R-algebra. Let $f\colon A \otimes_R A \to A$ be the multiplication homomorphism, i.e., $f(a \otimes b) = ab$, and let J be the kernel of f. Consider $A \otimes A$ as am A-module by multiplication on the left. Then J/J^2 inherits a structure of A-module. Define a map $d\colon A \to J/J^2$ by $da = 1 \otimes a - a \otimes 1$ (modulo J^2). Then $(J/J^2, d)$ is a universal derivation for A/R.*

Proposition 3.1.19 enables us to define a sheaf of relative differentials on a scheme X over a scheme Y. Using Remark 3.1.18 we will then give a similar characterization of this sheaf of differentials in the case that $Y = \operatorname{Spec} K$ for a field K.

Definition 3.1.20 *Let $f\colon X \to Y$ be any morphism of schemes and let $\Delta\colon X \to X \times_Y X$ be the diagonal morphism. Then Δ induces an isomorphism of X onto its image $\Delta(X)$, which is a closed subscheme of an open subset W of $X \times_Y X$ (see [47, prf. of Cor.II.4.2]). Let $\mathcal{I}$ be the sheaf of ideals of $\Delta(X)$ in W. Then we define the* sheaf of relative differentials *of X over Y to be the sheaf $\Omega_{X/Y} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$ on X.*

Remark 3.1.21 Let X be a scheme over a field K. From the characterization of the universal derivation in Remark 3.1.18 we find that the sheaf of differentials is the sheaf associated to the presheaf which can be characterized by

$$\Omega_{X/K}(U) \cong \Omega_{\mathcal{O}_X(U)/K}$$

for every open U of X. Note that the first Ω denotes a sheaf, where the second denotes an $\mathcal{O}_X(U)$-module.

Proposition 3.1.22 *Let X be an irreducible separated scheme of finite type over an algebraically closed field K. Then $\Omega_{X/K}$ is a locally free sheaf of rank $n = \dim X$ if and only if X is a nonsingular variety over K.*

Proof. See [47, Thm.II.8.15]. □

Definition 3.1.23 *Let X be a nonsingular variety over an algebraically closed field K and let $\Omega_{X/K}$ be the sheaf of differentials on X, then the* canonical sheaf ω_X *on X is defined as $\omega_X = \bigwedge^n \Omega_{X/K}$, the n-th exterior power of the sheaf of differentials.*

Let X be a nonsingular variety over an algebraically closed field K. The canonical sheaf ω_X corresponds to a divisor class K_X, which we call the *canonical divisor class*. Note that K without subscript denotes the field, while K_X denotes a divisor class. The constant sheaf $\mathcal{K}_X$ corresponds to the function field $K(X)$.

The sheaf ω_X and the class K_X are related as follows, see [47, Prop.II.6.15.] and [52, Ch.I.1.]. Choose a nonzero global section ω of ω_X, i.e., a differential form of degree $n = \dim X$. Note that Lang [52, p.14] forgets to take ω nonzero. For any closed point P on the nonsingular variety X let $t_1, \ldots, t_n$ be local parameters in P, i.e., generators for $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the maximal ideal of the local ring at P. Let U_P be a neighborhood of P on which for every point $Q \in U_P$ the $t_j - t_j(Q)$ form a set of local parameters. On U_P we may write $\omega = \psi_P dt_1 \wedge \ldots \wedge dt_n$ for some $\psi_P \in \mathcal{K}_X^*(U_P) = K(X)^*$. The collection of pairs (U_P, ψ_P) defines a Cartier divisor, which is called the *divisor associated with* ω and is denoted by (ω). Since the $K(X)$-vectorspace of differential forms of degree n has dimension 1 over $K(X)$, we find that all divisors associated to some nonzero differential form of degree n are linearly equivalent. They are called *canonical divisors*.

Conversely if D is a divisor which is linearly equivalent with the divisor (ω) associated with a differential ω, say $D = (f) + (\omega)$ for some rational function $f \in K(X)^*$, then D is the divisor associated with $f\omega$, so D is also a canonical divisor. The class of these divisors is the canonical divisor class K_X. By K_X we will sometimes also denote a divisor contained in the canonical class.

Since the divisor (ω) is defined by the pairs (U_P, ψ_P), we find that $\mathcal{L}((\omega))(U) = \psi_P^{-1}\mathcal{O}_X(U)$ for $U \subset U_P$. Hence every section of $\mathcal{L}((\omega))(U) \subset \mathcal{K}_X(U)$ comes from a rational function $f \in K(X)$ such that $f\psi_P$ is regular on U. Hence $f\omega$ is a regular differential form on U. This map

$$\mathcal{L}((\omega))(U) \to \omega_X(U)\colon f \mapsto f\omega$$

induces an isomorphism of sheaves between ω_X and the sub-$\mathcal{O}_X$-module $\mathcal{L}((\omega))$ of $\mathcal{K}_X$.

If Y is a nonsingular subvariety of a nonsingular variety X over an algebraically closed field K, then we can express the canonical sheaf on Y in the canonical sheaf of X and the so-called *normal sheaf* of Y in X, see Proposition 3.1.27 and [47, II.8]. We will first define this normal sheaf.

Proposition 3.1.24 *Let X be a nonsingular variety over an algebraically closed field K. Let $Y \subset X$ be an irreducible closed subscheme defined by a sheaf of ideals $\mathcal{I}$. Then Y is nonsingular if and only if $\Omega_{Y/K}$ is locally free and there is an exact sequence*

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{X/K} \otimes \mathcal{O}_Y \to \Omega_{Y/K} \to 0$$

of sheaves on Y. Furthermore, in this case, $\mathcal{I}$ is locally generated by $r = \operatorname{codim}(Y, X)$ elements, and $\mathcal{I}/\mathcal{I}^2$ is a locally free sheaf of rank r on Y.

Proof. See [47, Thm.II.8.17]. □

Definition 3.1.25 *Let Y be a closed subscheme of a scheme X over K and let $\mathcal{I}$ be the corresponding sheaf of ideals. We will call the sheaf $\mathcal{I}/\mathcal{I}^2$ on Y the* conormal *sheaf of Y in X. Its dual $\mathcal{N}_{Y/X} = \mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$ is called the* normal *sheaf of Y in X.*

Remark 3.1.26 Note that it follows from Proposition 3.1.24 that if Y is a nonsingular subvariety of a nonsingular variety X, then the normal sheaf $\mathcal{N}_{Y/X}$ is locally free of rank $r = \operatorname{codim}(Y, X)$. We will see later that it is also locally free if $X = \mathbb{P}^n$ and Y is a complete intersection.

Proposition 3.1.27 *Let Y be a nonsingular subvariety of codimension r in a nonsingular variety X. Let ω_Y and ω_X be the canonical sheaves on Y and X respectively. Then $\omega_Y \cong \omega_X \otimes \bigwedge^r \mathcal{N}_{Y/X}$. In case $r = 1$, consider Y as a divisor, and let $\mathcal{L} = \mathcal{L}(Y)$ be the associated invertible sheaf on X. Then $\omega_Y \cong \omega_X \otimes \mathcal{L} \otimes \mathcal{O}_Y$.*

Proof. See [47, Prop. II.8.20]. □

In some situations we can use Proposition 3.1.27 to identify the canonical sheaf on Y with a more familiar invertible sheaf. One specific such case is the situation in which Y is a *complete intersection* in $X = \mathbb{P}^n$. In that case we can combine Proposition 3.1.27 with Lemma 3.1.36.

Definition 3.1.28 *Let Y be a closed subscheme of a nonsingular variety X. We say that Y is a* local complete intersection *in X if the ideal sheaf $\mathcal{I}_Y$ of Y in X can be locally generated by $r = \operatorname{codim}(Y, X)$ elements at every point.*

Definition 3.1.29 *A closed subscheme Y of projective n-space $\mathbb{P}^n_K$ over a field K is called a* (strict,global) complete intersection *if the homogeneous ideal I of Y in $S = K[x_0, \ldots, x_n]$ can be generated by $r = \operatorname{codim}(Y, \mathbb{P}^n)$ elements.*

Complete intersections are closely related to *regular sequences*, whence to the notion *Cohen-Macaulay*. These notions are defined as follows.

Definition 3.1.30 *Let A be ring and M an A-module. Let $x_1, \ldots, x_r \in A$ elements such that*

(i) $M/(x_1, \ldots, x_r)M \neq 0$,

(ii) x_1 is not a zero divisor in M and for every $1 < j \leq r$ the element $\overline{x_j}$ is not a zero divisor in $M/(x_1, \ldots, x_{j-1})M$.

Then $x_1, \ldots, x_r$ is called a regular sequence *for M.*

Definition 3.1.31 *If A is a local ring with maximal ideal $\mathfrak{m}$, then the* depth *of M is the maximum length of a regular sequence $x_1, \ldots, x_r$ for M with all $x_j \in \mathfrak{m}$. We say that a local noetherian ring A is* Cohen-Macaulay *if $\operatorname{depth} A = \dim A$. A scheme X is said to be* Cohen-Macaulay *if all its local rings are Cohen-Macaulay.*

Proposition 3.1.32 *Let A be a local noetherian ring with maximal ideal $\mathfrak{m}$.*

(a) If A is regular, then it is Cohen-Macaulay.

(b) If A is Cohen-Macaulay, then every localization of A at a prime ideal is also Cohen-Macaulay.

(c) If A is Cohen-Macaulay, then a set of elements $x_1, \ldots, x_r \in \mathfrak{m}$ forms a regular sequence for A if and only if $\dim A/(x_1, x_2, \ldots, x_r) = \dim A - r$.

(d) If A is Cohen-Macaulay and $x_1, \ldots, x_r \in \mathfrak{m}$ is a regular sequence for A, then $A/(x_1, \ldots, x_r)$ is also Cohen-Macaulay.

(e) If A is Cohen-Macaulay, and $x_1, \ldots, x_r \in \mathfrak{m}$ is a regular sequence, let I be the ideal $I = (x_1, \ldots, x_r)$. Then the natural map

$$(A/I)[t_1, \ldots, t_r] \to \operatorname{gr}_I A = \bigoplus_{n \geq 0} I^n/I^{n+1},$$

defined by sending $t_j \mapsto x_j$, is an isomorphism. In particular, I/I^2 is a free A/I-module of rank r, generated by the x_j.

Proof. See [53, (a):p.121, (b,c,d):p.104–105, (e):p.110]. For the last statement of (e), see also [50, Prop.XXI.4.1]. □

Lemma 3.1.33 *Let K be an algebraically closed field and let H be a hypersurface in $\mathbb{P}^n_K$, i.e., a locally principal subscheme of codimension 1. Then there is a homogeneous principal ideal $I \subset K[x_0, \ldots, x_n]$, such that I is the homogeneous ideal of H. In other words, H is a complete intersection.*

Proof. Since H is locally principal, we can view H as an effective Cartier divisor and as $\mathbb{P}^n$ is a noetherian regular variety, even as an effective Weil divisor. If divisors D_1 and D_2 have principal homogeneous ideals (f_1) and (f_2) respectively, then the divisor $D_1 + D_2$ has principal ideal $(f_1 f_2)$, so we may assume that H is a prime Weil divisor. Let l be a linear homogeneous form such that H is not contained in the hypersurface $l = 0$ and let U be the open affine given by $l \neq 0$. By a homogeneous transformation of coordinates we may assume that $l = x_0$. As $H \cap U$ is an irreducible hypersurface in $U = \operatorname{Spec} K[x_1/x_0, \ldots, x_n/x_0] \cong \mathbb{A}^n$, we find that $H \cap U$ is in U the zeroset of an irreducible polynomial f in $K[x_1/x_0, \ldots, x_n/x_0]$ (see [47, Prop.I.1.13]). Let F be the homogenised polynomial of f with respect to x_0, i.e., $F = x_0^{\deg f} \cdot f$. Then $I = (F) \subset K[x_0, \ldots, x_n]$ is the homogeneous ideal of the closure of $H \cap U$ in $\mathbb{P}^n$. This is exactly H itself. □

Corollary 3.1.34 *Let K be an algebraically closed field and let Y be a closed subscheme of codimension r in $\mathbb{P}^n_K$. Then Y is a complete intersection if and only if there are hypersurfaces (i.e., locally principal subschemes of codimension 1) $H_1, \ldots, H_r$, such that $Y = H_1 \cap \cdots \cap H_r$* as schemes, *i.e., $\mathcal{I}_Y = \mathcal{I}_{H_1} + \ldots + \mathcal{I}_{H_r}$.*

Proof. Let Y be a complete intersection of codimension r given by the homogeneous ideal $I = (f_1, \ldots, f_r)$. The f_j all determine a closed subvariety H_j. The H_j are locally principal for if l is a linear homogeneous form, then on the affine open set given by $l \neq 0$ the ideal sheaf of the hypersurface H_j is generated by f_j/l^{d_j}, where d_j is the degree of f_j. Further, since $\mathbb{P}^n$ is integeral, the f_j/l^{d_j} are not zero divisors in the local rings at points of $\mathbb{P}^n$. As $\mathbb{P}^n$ is also Cohen-Macaulay, it follows that the H_j have codimension 1. Hence the H_j are hypersurfaces. On the same affine open $l \neq 0$ the ideal sheaf of Y is generated by $f_1/l^{d_1}, \ldots, f_r/l^{d_r}$. As these open affines cover $\mathbb{P}^n$ we find $\mathcal{I}_Y = \mathcal{I}_{H_1} + \ldots + \mathcal{I}_{H_r}$.

Conversely, let $H_1, \ldots, H_r$ be locally principal subschemes of codimension 1 in $\mathbb{P}^n$, such that $\mathcal{I}_Y = \mathcal{I}_{H_1} + \ldots + \mathcal{I}_{H_r}$. Then Lemma 3.1.33 states that there are principal homogeneous ideals $I_j = (f_j)$, such that H_j is the hypersurface given by $f_j = 0$, i.e., on the open affine $l \neq 0$ the ideal sheaf $\mathcal{I}_{H_j}$ is generated by f_j/l^{d_j}. That means that on the same open affine the ideal sheaf $\mathcal{I}_Y$ is generated by $f_1/l^{d_1}, \ldots, f_r/l^{d_r}$. This implies that the homogeneous ideal I of Y is $I = (f_1, \ldots, f_r)$ and can be generated by r elements. Therefore Y is a complete intersection. See also [47, exc.II.8.4(a)]. □

Remark 3.1.35 Note that a strict complete intersection $Y \subset \mathbb{P}^n$ is indeed a local complete intersection as well, since the H_j of which Y is the intersection are locally principal.

Lemma 3.1.36 *Let K be an algebraically closed field and let $\rho\colon Y \hookrightarrow \mathbb{P}^n_K$ be a global complete intersection of codimension r, say of hypersurfaces $D_1, \ldots, D_r$ of degree $d_1, d_2, \ldots, d_r$ respectively. Let $\mathcal{N}_{Y/\mathbb{P}^n}$ be the normal sheaf of Y in $\mathbb{P}^n$. Let $\omega_{\mathbb{P}}$ be the canonical sheaf of $\mathbb{P}^n$. Then we have $\omega_{\mathbb{P}} \cong \mathcal{O}_{\mathbb{P}}(-n-1)$ and if we set $m = -n - 1 + \sum d_i$, then*

$$\omega_{\mathbb{P}} \otimes \bigwedge^r \mathcal{N}_{Y/\mathbb{P}^n} \cong \rho^* \mathcal{O}_{\mathbb{P}}(m).$$

Before we prove this, first some algebraic results. The following result about exterior powers will be needed.

Lemma 3.1.37 *Let*

$$0 \to E' \to E \to E'' \to 0$$

be an exact sequence of free R-modules of finite rank r, n and s respectively. Then there is a natural isomorphism

$$\varphi\colon \bigwedge^r E' \otimes \bigwedge^s E'' \to \bigwedge^n E.$$

This isomorphism is the unique isomorphism having the following property. For the n elements $v_1, \ldots, v_r \in E'$ and $w_1, \ldots, w_s$ let $u_1, \ldots u_s$ be liftings of $w_1, \ldots, w_s$ in E. Then

$$\varphi((v_1 \wedge \cdots \wedge v_r) \otimes (w_1 \wedge \cdots \wedge w_s)) = v_1 \wedge \cdots \wedge v_r \wedge u_1 \wedge \cdots \wedge u_s.$$

Proof. See [50, Prop.XIX.1.2.]. □

In manipulating modules the following lemma will turn out to be quite useful.

Lemma 3.1.38 *Let R and S be rings and let A_R, ${}_RB_S$ and C_S be (bi-)modules. Then there is a natural isomorphism*

$$\alpha\colon \mathrm{Hom}_S(A \otimes_R B, C) \to \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$$

defined for each $f\colon A \otimes_R B \to C$ by

$$[(\alpha f)(a)](b) = f(a \otimes b).$$

Proof. See [49, Thm.IV.5.10]. □

Lemma 3.1.39 *Let A be a local noetherian ring which is Cohen-Macaulay and suppose that the elements $x_1, \ldots, x_r \in \mathfrak{m}$ form a regular sequence for A. Let $I_j \subset A$ denote the ideal $I_j = (x_j)$ and let I denote the ideal $I = (x_1, \ldots, x_r)$. Then I_j/I_j^2 is an A/I_j-module and I/I^2 is an A/I-module.*

The natural maps $I_j/I_j^2 \to I/I^2$ induce natural maps $I_j/I_j^2 \otimes A/I \to I/I^2$ which induce a natural homomorphism

$$f\colon \bigoplus_{j=1}^{r} \left(I_j/I_j^2 \otimes_{A/I_j} A/I\right) \to I/I^2 \tag{16}$$

of A/I-modules. This homomorphism is actually an isomorphism. It induces the following natural isomorphism.

$$\operatorname{Hom}_{A/I}(I/I^2, A/I) \to \bigoplus_{j=1}^{r} \left(\operatorname{Hom}_{A/I_j}(I_j/I_j^2, A/I_j) \otimes_{A/I_j} A/I\right). \tag{17}$$

Proof. From Proposition 3.1.32 we know that I_j/I_j^2 is a free A/I_j-module generated by $\overline{x_j}$, so $I_j/I_j^2 \otimes A/I$ is a free A/I-module of rank 1, generated by $\overline{x_j} \otimes 1$. The A/I-module I/I^2 is by the same proposition also a free A/I-module, generated by $\overline{x_1}, \ldots, \overline{x_r}$ and of rank r. Hence the two homomorphisms

$$\varphi\colon (A/I)^r \to I/I^2\colon (a_j)_{j=1}^r \mapsto \sum_{j=1}^{r} a_j \overline{x_j},$$

$$\psi\colon (A/I)^r \to \bigoplus_{j=1}^{r} \left(I_j/I_j^2 \otimes A/I\right) : (a_j)_{j=1}^r \mapsto (\overline{x_j} \otimes a_j)_{j=1}^r$$

are both isomorphisms. These isomorphisms make the following commutative diagram, from which it follows that f is an isomorphism as well.

$$\begin{array}{ccc} \bigoplus_{j=1}^r \left(I_j/I_j^2 \otimes A/I\right) & \xrightarrow{\ f\ } & I/I^2 \\ & {}_{\psi}\nwarrow^{\sim} \quad {}^{\sim}\nearrow_{\varphi} & \\ & (A/I)^r & \end{array}$$

Using the fact that f is an isomorphism we find a sequence of canonical isomorphisms.

$$\begin{aligned} \operatorname{Hom}_{A/I}\left(I/I^2, A/I\right) &\cong \operatorname{Hom}_{A/I}\left(\bigoplus_{j=1}^{r}(I_j/I_j^2 \otimes_{A/I_j} A/I), A/I\right) \\ &\cong \bigoplus_{j=1}^{r} \operatorname{Hom}_{A/I}\left(I_j/I_j^2 \otimes_{A/I_j} A/I, A/I\right) \\ &\cong \bigoplus_{j=1}^{r} \operatorname{Hom}_{A/I_j}\left(I_j/I_j^2, \operatorname{Hom}_{A/I}(A/I, A/I)\right) \\ &\cong \bigoplus_{j=1}^{r} \operatorname{Hom}_{A/I_j}\left(I_j/I_j^2, A/I\right) \\ &\cong \bigoplus_{j=1}^{r} \operatorname{Hom}_{A/I_j}\left(I_j/I_j^2, A/I_j\right) \otimes_{A/I_j} A/I. \end{aligned}$$

The first isomorphism follows from the fact that f is an isomorphism, the second from the fact that a finite direct sum is a product. The third follows from Lemma 3.1.38 and the fourth since $\operatorname{Hom}_R(R, R) \cong R$. Finally the fifth isomorphism follows from the fact that I_j/I_j^2 is a free A/I_j-module of rank 1. This isomorphism is therefore induced by the natural homomorphim of A/I_j-modules

$$\operatorname{Hom}_{A/I_j}(I_j/I_j^2, A/I_j) \to \operatorname{Hom}_{A/I_j}(I_j/I_j^2, A/I).$$

□

Lemma 3.1.39 is purely algebraic. The sheaves version of it will enable us to prove Lemma 3.1.36.

Corollary 3.1.40 *Let K be an algebraically closed field and let Y be a closed subscheme of codimension $r \leq n$ in $X = \mathbb{P}^n_K$. Suppose that Y is a global complete intersection, say given as the intersection $D_1 \cap \cdots \cap D_r$, where the D_j are hypersurfaces in $\mathbb{P}^n$. Let $\mathcal{I}_j$ be the ideal sheaf of D_j in X and let $\mathcal{I}_Y$ be the ideal sheaf of Y in X. Then the natural morphism of sheaves on Y*

$$\bigoplus_{j=1}^{r} \left(\mathcal{I}_j/\mathcal{I}_j^2 \otimes_{\mathcal{O}_{D_j}} \mathcal{O}_Y \right) \to \mathcal{I}_Y/\mathcal{I}_Y^2 \tag{18}$$

is an isomorphism. It induces a natural isomorphism of sheaves

$$\mathcal{N}_{Y/X} \cong \bigoplus_{j=1}^{r} \mathcal{N}_{D_j/X} \otimes \mathcal{O}_Y, \tag{19}$$

where $\mathcal{N}_{Y/X}$ and $\mathcal{N}_{D_j/X}$ denote the normal sheaves of Y and D_j in X respectively.

Proof. Note that the natural morphism $\mathcal{I}_j/\mathcal{I}_j^2 \to \mathcal{I}_Y/\mathcal{I}_Y^2$ indeed induces a natural morphism as in (18). It suffices to check that it is an isomorphism locally. Let P be a closed point on X. Since a global complete intersection is also a local complete intersection there is an open affine neighborhood $U \subset \mathbb{P}^n$ of P over which $\mathcal{I}_Y$ can be generated by r elements $f_1, \ldots, f_r \in B = \Gamma(U, \mathcal{O}_U)$. Let $\mathfrak{p} \subset B$ be the maximal ideal corresponding to the point $P \in U \cong \operatorname{Spec} B$. Since X is nonsingular, the local ring $A = \mathcal{O}_{P,X} \cong B_{\mathfrak{p}}$ is regular, whence Cohen-Macaulay by Proposition 3.1.32. The ideal I_Y of Y in $\operatorname{Spec} A$ is given by $I_Y = (f_1, \ldots, f_r)$ and as Y is of codimension r in U we have $\dim A/I_Y = \dim A - r$, so from the same Proposition 3.1.32 we find that $f_1, \ldots, f_r$ is a regular sequence for A. The ideal I_j of D_j in $\operatorname{Spec} A$ is given by $I_j = (f_j)$, so on $\operatorname{Spec} A$ the morphism (18) is given by

$$\bigoplus_{j=1}^{r} \left(I_j/I_j^2 \otimes_{A/I_j} A/I \right) \to I_Y/I_Y^2,$$

by taking the corresponding sheaves of modules. This is exactly the isomorphism (16) of Lemma 3.1.39 and hence proves that (18) is an isomorphism. Similarly, it follows locally from the isomorphism (17) of Lemma 3.1.39 that the homomorphism in (19) is an isomorphism. □

Proof of Lemma 3.1.36. For the first assertion, see [47, exa.II.8.20.1]. Let $\mathcal{I}$ and $\mathcal{I}_j$ be the sheaves of ideals of Y and D_j respectively in $\mathbb{P}^n$. From Lemma 3.1.12 we find $\mathcal{I}_j \cong \mathcal{L}(-D_j)$ and hence the sheaf $\mathcal{I}_j/\mathcal{I}_j^2$ on D_j is given by $\mathcal{L}(-D_j) \otimes \mathcal{O}_{D_j}$. Since D_j is as a divisor on $\mathbb{P}^n$ linearly equivalent with $d_j H$, where H is any hyperplane, it follows that we have an isomorphism

$$\mathcal{I}_j/\mathcal{I}_j^2 \cong \mathcal{L}(-D_j) \otimes \mathcal{O}_{D_j} \cong \mathcal{L}(-d_j H) \otimes \mathcal{O}_{D_j} \cong \mathcal{O}_{\mathbb{P}}(-d_j) \otimes \mathcal{O}_{D_j} \tag{20}$$

of invertible sheaves on D_j. We get the following sequence of isomorphisms.

$$\begin{aligned}
\mathcal{N}_{Y/\mathbb{P}^n} &\cong \mathcal{H}\mathrm{om}_{\mathcal{O}_Y}\left(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y\right) \\
&\cong \mathcal{H}\mathrm{om}_{\mathcal{O}_Y}\left(\bigoplus_{j=1}^{r}\left(\mathcal{I}_j/\mathcal{I}_j^2 \otimes \mathcal{O}_Y\right), \mathcal{O}_Y\right) \\
&\cong \bigoplus_{j=1}^{r} \mathcal{H}\mathrm{om}_{\mathcal{O}_Y}\left(\mathcal{O}_{\mathbb{P}}(-d_j) \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_Y, \mathcal{O}_Y\right) \\
&\cong \bigoplus_{j=1}^{r} \mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathbb{P}}}\left(\mathcal{O}_{\mathbb{P}}(-d_j), \mathcal{H}\mathrm{om}_{\mathcal{O}_Y}\left(\mathcal{O}_Y, \mathcal{O}_Y\right)\right) \\
&\cong \bigoplus_{j=1}^{r} \mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathbb{P}}}\left(\mathcal{O}_{\mathbb{P}}(-d_j), \mathcal{O}_Y\right) \\
&\cong \bigoplus_{j=1}^{r} \mathcal{H}\mathrm{om}_{\mathcal{O}_{\mathbb{P}}}\left(\mathcal{O}_{\mathbb{P}}(-d_j), \mathcal{O}_{\mathbb{P}}\right) \otimes \mathcal{O}_Y \\
&\cong \bigoplus_{j=1}^{r} \mathcal{O}_{\mathbb{P}}(d_j) \otimes \mathcal{O}_Y.
\end{aligned}$$

The first isomorphism is a definition and the second follows from the isomorphism (18) of Corollary 3.1.40. The third follows from (20) and the fact that a finite direct sum is a direct product. The fourth isomorphism follows from the sheaves version of Lemma 3.1.38, while the fifth is trivial. The sixth follows from the fact that $\mathcal{O}_{\mathbb{P}}(d)$ is a locally free $\mathcal{O}_{\mathbb{P}}$-module of rank 1. Finally the last isomorphism follows from the fact that for any invertible sheaf $\mathcal{L}$ on a scheme X we have $\mathcal{L}^{-1} \cong \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$.

Since $\mathcal{O}_{\mathbb{P}}(d) \otimes \mathcal{O}_Y$ is locally free of rank 1, we have $\bigwedge^1 \mathcal{O}_{\mathbb{P}}(d) \otimes \mathcal{O}_Y = \mathcal{O}_{\mathbb{P}}(d) \otimes \mathcal{O}_Y$, so by repeated use of Lemma 3.1.37 we find an isomorphism

$$\begin{aligned}
\bigwedge^{r} \mathcal{N}_{Y/\mathbb{P}^n} &\cong \bigwedge^{r} \bigoplus_{j=1}^{r} \mathcal{O}_{\mathbb{P}}(d_j) \otimes \mathcal{O}_Y \\
&\cong \mathcal{O}_{\mathbb{P}}(d_1) \otimes \cdots \otimes \mathcal{O}_{\mathbb{P}}(d_r) \otimes \mathcal{O}_Y \\
&\cong \mathcal{O}_{\mathbb{P}}\left(\sum d_j\right) \otimes \mathcal{O}_Y.
\end{aligned}$$

Using $\omega_{\mathbb{P}} \cong \mathcal{O}_{\mathbb{P}}(-n-1)$ we find

$$\omega_{\mathbb{P}} \otimes \bigwedge^{r} \mathcal{N}_{Y/\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}}(-n-1) \otimes \mathcal{O}_{\mathbb{P}}\left(\sum d_j\right) \otimes \mathcal{O}_Y \cong \mathcal{O}_{\mathbb{P}}(m) \otimes \mathcal{O}_Y = \rho^* \mathcal{O}_{\mathbb{P}}(m).$$

□

Two notions defined on nonsingular varieties in terms of the canonical sheaf are the m-genus p_m and the geometrical genus p_g.

Definition 3.1.41 *For a positive integer m and a nonsingular projective variety X over an algebraically closed field K the m-genus is defined as*

$$p_m(X) = \dim_K H^0(X, \omega_X^{\otimes m}).$$

The geometrical genus p_g is defined as $p_g(X) = p_1(X)$.

These notions are birational invariants, see [47, Thm.II.8.19, Exc.II.8.8] and [60, Cor.II.6.4]. Hence in characteristic 0 we can define the m-genus and the geometric genus of an arbitrary variety to be the corresponding genus of a birational nonsingular projective variety due to resolution of singularities by Hironaka.

There are some other numbers that people seem to be interested in. Instead of the canonical sheaf ω_X we can also use the sheaf of differentials $\Omega_{X/K}$ to define the numbers $h^{p,q}$ as in Definition 3.1.42 for every pair (p, q) of nonnegative integers. These are in general not birational invariants.

Definition 3.1.42 *Let X be a nonsingular connected compact variety over an algebraically closed field K of dimension n and let $\Omega_{X/K}$ be its sheaf of differentials. For integers p, q the group $H^{p,q}(X)$ and the number $h^{p,q}(X)$ are defined as*

$$H^{p,q}(X) = H^q(X, \bigwedge^p \Omega_{X/K}) \qquad \textit{and} \qquad h^{p,q}(X) = \dim_K H^{p,q}(X).$$

The numbers $h^{p,q}$ are called the Hodge numbers*.*

Remark 3.1.43 Note that from the fact that $\omega_X = \bigwedge^n \Omega_X$ we find that $p_g(X) = h^{n,0}(X)$.

Proposition 3.1.44 (Grothendieck's Vanishing Theorem) *Let X be a noetherian topological space of dimension n. Then for all $i > n$ and all sheaves of abelian groups $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F}) = 0$.*

Proof. See [47, Thm.III.2.7]. □

Corollary 3.1.45 *Let X be a noetherian nonsingular connected compact variety of dimension n. For nonnegative integers p, q with $p > n$ or $q > n$ we have $h^{p,q}(X) = 0$.*

Proof. If $q > n$, then it follows from Proposition 3.1.44 that $H^q(X, \bigwedge^p \Omega_X) = 0$. If $p > n$ then it follows from the fact that $\bigwedge^p \Omega_X = 0$. □

If X is defined over the complex numbers $\mathbb{C}$, then we can view $X = X(\mathbb{C})$ as a connected compact complex manifold of dimension n. Hence we can consider the cohomology groups $H^k(X, \mathbb{C})$ of X with coefficients in $\mathbb{C}$.

Definition 3.1.46 *Let X be a connected compact complex manifold. Then we define the k-th* Betti number *$b_k(X)$ to be $b_k(X) = \dim_{\mathbb{C}} H^k(X, \mathbb{C})$.*

Serre's paper GAGA [55] tells us that we can identify $H^{p,q}(X) = H^q(X, \bigwedge^p \Omega_X)$ with its analytic brother $H^q(X_h, \bigwedge^p \Omega_{X_h})$, where Ω_{X_h} is the sheaf of holomorphic differential forms on the analytic space X_h associated to X, see also [47, Thm.B.2.1]. Under this identification we get the following proposition.

Proposition 3.1.47 *Let X be a a nonsingular connected projective variety defined over $\mathbb{C}$. Then there is a direct sum decomposition*

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X),$$

which is the famous Hodge decomposition*. Hence we have $b_k = \sum_{p+q=k} h^{p,q}$. We also have $h^{p,q} = h^{q,p}$.*

Proof. See [35, Cor's.I.13.3,4]. Note that these corollaries in [35] are stated for compact Kähler manifolds, but projective algebraic manifolds are indeed Kählerian. Note also that the definition of $H^{p,q}(X)$ is different in [35], but from the Dolbeault isomorphism from [35, Sect.I.12] we see that their definition coincides with ours. □

Proposition 3.1.48 (Poincaré duality) *If X is a nonsingular projective variety of dimension n defined over $\mathbb{C}$, then* Poincaré duality *gives isomorphisms*

$$H^k(X, \mathbb{C}) \cong H^{2n-k}(X, \mathbb{C})$$

for $1 \leq k \leq 2n$, whence $b_k(X) = b_{2n-k}(X)$.

Proof. See [35, Sec.I.1.]. □

Remark 3.1.49 Note that for $k > 2n$ we have $b_k = 0$. This is a general result from algebraic topology, but also follows from Proposition 3.1.47 and Corollary 3.1.45. Note also that for a projective nonsingular surface over an algebraically closed field K we have $H^0(X, \Omega_X^0) = H^0(X, \mathcal{O}_X) = K$, so $b_0(X) = h^{0,0}(X) = \dim_K K = 1$. From Proposition 3.1.48 we also find $b_{2n}(X) = h^{n,n}(X) = 1$ if $K = \mathbb{C}$.

The betti and Hodge numbers are related by the equation $b_n = \sum_{p+q=n} h^{p,q}$, but there are more relations. One of these can be best expressed by two different Euler characteristics, the topological Euler characteristic and the Euler characteristic of the global sections.

Definition 3.1.50 *Let X be a connected compact complex manifold. Then we define the* topological Euler characteristic *as*

$$\chi_{\text{top}}(X) = \sum_{k=1}^{2n} (-1)^k \dim H^k(X, \mathbb{C}).$$

Definition 3.1.51 *Let X be a projective variety over a field K of dimension n. As the sheaf $\mathcal{O}_X$ is coherent, the groups $H^k(X, \mathcal{O}_X)$ are finite dimensional, so we can define the* Euler characteristic (of global sections) *by*

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \dim_K H^k(X, \mathcal{O}_X)$$

of X and the arithmetic genus $p_a(X) = (-1)^n(\chi(X) - 1)$.

Proposition 3.1.52 (Noether formula) *Let X be a nonsingular projective variety of dimension n defined over $\mathbb{C}$ and let K_X be a canonical divisor on X. Then $\chi(X)$ and $\chi_{\text{top}}(X(\mathbb{C}))$ are related by*

$$\chi_{\text{top}}(X(\mathbb{C})) = 12\chi(X) - K_X^2.$$

Proof. See [36, Sect.I.14, p.12]. □

Lemma 3.1.53 *Let X be a nonsingular projective variety over an algebraically closed field K and let ω_X be the canonical sheaf. Then we can also express the Euler characteristic as*

$$\chi(X) = \sum_{k=0}^{n} (-1)^k \dim_K H^{n-k}(X, \omega_X) = \sum_{k=0}^{n} (-1)^k h^{n,n-k}(X).$$

Proof. Serre's duality (see [35, Thm.I.5.2] or [47, Cor.III.7.7]) says that if X is a nonsingular variety with canonical sheaf ω_X, then there is an isomorphism $H^k(X, \mathcal{O}_X) \cong H^{n-k}(X, \omega_X)$. The first equality now follows directly. The second follows from the definition of the Hodge numbers and the fact that $\bigwedge^n \Omega_X \cong \omega_X$. □

For curves and surfaces the arithmetic genus is a birational invariant. In general it is invariant under *monoidal transformations*, which comes close.

Definition 3.1.54 *A* monoidal transformation *of a variety X is the operation of blowing up a single point P.*

Lemma 3.1.55 *If $\widetilde{X} \to X$ is a monoidal transformation, then there are isomorphisms*

$$H^i(X, \mathcal{O}_X) \cong H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$$

for all $i \geq 0$.

Lemma 3.1.56 *Let $\pi \colon \widetilde{X} \to X$ be a monoidal transformation. Then $p_a(\widetilde{X}) = p_a(X)$. If X is either a nonsingular projective curve or a nonsingular projective surface, then $p_a(X)$ and $\chi(X)$ are birational invariants.*

Proof. For the first claim see [47, Cor.V.3.5]. For the last two see [47, Exc.II.5.3] for the curve case and [47, Cor.V.5.6] for the surface. □

Remark 3.1.57 For a curve C we find from Lemma 3.1.53 with $n = 1$ and the fact that $h^{1,1}(C) = 1$ that

$$p_a(C) = 1 - \chi(C) = 1 - \big(h^{1,1}(C) - h^{1,0}(C)\big) = h^{1,0}(C) = p_g(C).$$

The number $p_g(C) = p_a(C)$ is called the genus of C. For a surface S we find similarly that

$$p_a(S) = p_g(S) - h^{2,1}(S) \leq p_g(S).$$

Lemma 3.1.58 *Let Y be a closed subvariety of $\mathbb{P}^n = \mathbb{P}^n_K$ defined over a field K with homogeneous coordinate ring $S(Y) = \bigoplus_{d \geq 0} S_d$, where S_d is the degree d part of $S(Y)$. Then there is a unique polynomial P_Y of degree $\dim Y$ such that $P_Y(l) = \dim_K S_l$ for l large enough.*

Proof. See [47, Thm. I.7.5]. □

Definition 3.1.59 *The polynomial mentioned in Lemma 3.1.58 is called the* Hilbert polynomial *P_Y of the projective variety Y.*

Lemma 3.1.60 *Let $Y \subset \mathbb{P}^n$ be a projective variety with Hilbert polynomial P_Y. Then for the Euler characteristic $\chi(Y)$ we have $\chi(Y) = P_Y(0)$.*

Proof. See [47, exc.III.5.3]. □

3.2 A surface describing perfect cuboids

A perfect cuboid corresponds to a rational point on the surface $\Upsilon \subset \mathbb{P}^6_{\mathbb{Q}}$ given by the equations

$$\begin{aligned} A^2 + B^2 - Z^2 &= 0, \\ B^2 + C^2 - X^2 &= 0, \\ C^2 + A^2 - Y^2 &= 0, \\ A^2 + X^2 - U^2 &= 0. \end{aligned} \tag{21}$$

Lemma 3.2.1 *Let K be any field of characteristic $\operatorname{char} K \neq 2$ and consider the graded ring $R = K[A, B, C, X, Y, Z, U]$. Then the ideal $I \subset R$ generated by the polynomials in (21) is a prime ideal.*

Proof. We will prove this lemma by showing that R/I has no zero divisors. Consider the polynomial ring $S = K(A, B, C)[X, Y, Z, U]$ in four variables over the field $L = K(A, B, C)$. We will first show that R/I is a subring of S/IS, and then that S/IS is a field. These two facts indeed imply that R/I has no zero divisors.

For the first fact, namely that R/I is a subring of S/IS, it suffices to show that I is the kernel of the natural map $R \to S/IS$, i.e., that $R \cap IS = I$. We use the theory of Gröbner bases to show this. The ordering $X > Y > Z > U$ induces a well-ordered lexicographic ordering on the monomials of the polynomial ring S. One can check that the four polynomials of (21) give a Gröbner basis with for IS with respect to this ordering.

Now suppose that for some $f \in R \subset S$ we have $f \in IS$. Since the monomials are well-ordered we can use induction with respect to the leading monomial of f to show that $f \in I$. For $f = 0$ this is immediate, so suppose that we have a nonzero $f \in R \cap IS$. Since the polynomials of (21) give a Gröbner basis for IS, the leading monomial M of f is divisible by the leading monomial of one of the polynomials of (21), whence by X^2, Y^2, Z^2 or U^2. Suppose M is divisible by X^2, then we can define

$$g = f - \frac{cM}{X^2} \cdot (X^2 - B^2 - C^2),$$

where $c \in L$ is the coefficient of M in f. Since we have $f \in R$, the leading coefficient c is contained in $K[A, B, C]$, whence $cM/X^2 \in R$. We find $g \in R \cap IS$ and since the leading monomial of g is less than M, by the induction hypothesis we may assume that $g \in I$ and it follows that $f \in I$. The cases that M is divisible by Y^2, Z^2 or U^2 are similar, so we find indeed that $R \cap IS = I$ and $R/I \subset S/IS$. Define

$$\begin{aligned}
L_0 &= L = K(A, B, C), && \\
L_1 &= L_0[X]/(X^2 - a_1), && \text{with} \quad a_1 = B^2 + C^2, \\
L_2 &= L_1[Y]/(Y^2 - a_2), && \text{with} \quad a_2 = A^2 + C^2, \\
L_3 &= L_2[Z]/(Z^2 - a_3), && \text{with} \quad a_3 = A^2 + B^2, \\
L_4 &= L_3[U]/(U^2 - a_4), && \text{with} \quad a_4 = A^2 + B^2 + C^2.
\end{aligned}$$

From Kummer theory we know that if the sequence

$$L_0^{*2} \subset L_0^{*2} \cdot \langle a_1 \rangle \subset L_0^{*2} \cdot \langle a_1, a_2 \rangle \subset L_0^{*2} \cdot \langle a_1, a_2, a_3 \rangle \subset L_0^{*2} \cdot \langle a_1, a_2, a_3, a_4 \rangle \qquad (22)$$

of subgroups of L_0^* is strictly increasing, then we will have $a_j \notin L_{j-1}$ for all $1 \le j \le 4$, whence L_j would be a field for all j. In particular, $L_4 \cong S/IS$ would be a field. It remains to show that the sequence (22) is indeed stricly increasing.

We will use the fact that $K[A, B, C]$ is a unique factorization domain. Consider the prime element $B + iC$, where $i^2 = -1$. Every element in L_0^{*2} has an even number of factors $B + iC$. Since a_1 doesn't, we find $a_1 \notin L_0^{*2}$. Similarly every element in $L_0^{*2} \cdot \langle a_1 \rangle$ has an even number of factors $A + iC$, so $a_2 \notin L_0^{*2} \cdot \langle a_1 \rangle$ and analogously $a_3 \notin L_0^{*2} \cdot \langle a_1, a_2 \rangle$. For the last step note that a_4 itself is prime in $K[A, B, C]$, for if we view a_4 as a polynomial in the variable C over the unique factorization domain $K[A, B]$, then a_4 is an Eisenstein polynomial with respect to the prime $A + iB$. Since a_1, a_2 and a_3 are not divisible by a_4, every element in $L_0^{*2} \cdot \langle a_1, a_2, a_3 \rangle$ has an even number of factors a_4. Therefore this group does not contain a_4 and the sequence (22) is strictly increasing. □

From Lemma 3.2.1 and the fact that $\dim \Upsilon = 2$ it follows that the surface Υ is a complete intersection. It is geometrically irreducible and reduced, whence integral, has degree 16 and its radical ideal $I(\Upsilon)$ in $\mathbb{Q}[A, B, C, X, Y, Z, U]$ is given by the polynomials in (21). We will denote the homogeneous coordinate ring of Υ by $S(\Upsilon) \cong \mathbb{Q}[A, B, C, X, Y, Z, U]/I(\Upsilon)$.

The map σ that sends a point $[a : b : c : x : y : z : u]$ to $[b : a : c : y : x : z : u]$ and the map $\tau : [a : b : c : x : y : z : u] \mapsto [b : c : a : y : z : x : u]$ are automorphisms of Υ. For t one of the seven coordinates let ι_t denote the involution that multiplies the t-coordinate with -1. These nine automorphisms generate a Galois invariant subgroup G_0 of $\mathrm{Aut}(\Upsilon)$.

The subgroup N generated by the ι_t is normal in G_0. Since the only relation among the ι_t is given by $\iota_A \iota_B \cdots \iota_U = 1$, the group N is generated by $\iota_A, \iota_B, \iota_C, \iota_X, \iota_Y, \iota_Z$ and isomorphic to $(\mathbb{Z}/2)^6$ under the isomorphism

$$\varphi \colon N \to (\mathbb{Z}/2)^6 \colon \iota_A^{q_A} \iota_B^{q_B} \iota_C^{q_C} \iota_X^{q_X} \iota_Y^{q_Y} \iota_Z^{q_Z} \iota_U^{q_U} \mapsto (q_A - q_U, q_B - q_U, q_C - q_U, q_X - q_U, q_Y - q_U, q_Z - q_U).$$

The subgroup $H = \langle \sigma, \tau \rangle$ is isomorphic to S_3. The subgroup H acts on N by conjugation. The corresponding action on $(\mathbb{Z}/2)^6$ via φ is given by letting S_3 act on the first and the last three coordinates at the same time. We have $HN = G_0$ and $H \cap N = \{1\}$, so G_0 is the semidirect product of H and N of order $6 \cdot 2^6 = 384$. Hence we have $G_0 \cong S_3 \ltimes (\mathbb{Z}/2)^6$ with respect to the action given by letting S_3 act on the first and the last 3 coordinates at the same time.

Consider a point $P = [a_0 : b_0 : c_0 : x_0 : y_0 : z_0]$ of $\Upsilon \otimes \overline{\mathbb{Q}}$ with $a_0, z_0 \neq 0$ with local ring $\mathcal{O}_{P,\Upsilon\otimes\overline{\mathbb{Q}}}$ and maximal ideal $\mathfrak{m}$. We can write

$$\frac{Z}{A} - \frac{z_0}{a_0} = \frac{a_0}{2z_0}\left(\frac{B}{A} - \frac{b_0}{a_0}\right)^2 + \frac{b_0}{z_0}\left(\frac{B}{A} - \frac{b_0}{a_0}\right) - \frac{a_0}{2z_0}\left(\frac{Z}{A} - \frac{z_0}{a_0}\right)^2,$$

so the element $\frac{Z}{A} - \frac{z_0}{a_0} \in \mathfrak{m}$ is congruent to $\frac{b_0}{z_0}(\frac{B}{A} - \frac{b_0}{a_0})$ modulo $\mathfrak{m}^2$. Similarly, if $x_0, y_0, u_0 \neq 0$, then we can express the residues of $\frac{X}{A} - \frac{x_0}{a_0}$, $\frac{Y}{A} - \frac{y_0}{a_0}$ and $\frac{U}{A} - \frac{u_0}{a_0}$ in $\mathfrak{m}/\mathfrak{m}^2$ in those of $\frac{B}{A} - \frac{b_0}{a_0}$ and $\frac{C}{A} - \frac{c_0}{a_0}$, whence the vector space $\mathfrak{m}/\mathfrak{m}^2$ is generated by $\frac{B}{A} - \frac{b_0}{a_0}$ and $\frac{C}{A} - \frac{c_0}{a_0}$ and therefore 2-dimensional, so P is regular. Analogously, we can define the following three open subsets of $\Upsilon \otimes \overline{\mathbb{Q}}$ in which every point is regular.

open set	given by	local parameters
U_1	$X, Y, Z, A, B \neq 0$	$\frac{C}{A} - \frac{c_0}{a_0}, \frac{U}{A} - \frac{U_0}{a_0}$
U_2	$U, X, A, B, C \neq 0$	$\frac{Y}{A} - \frac{y_0}{a_0}, \frac{Z}{A} - \frac{z_0}{a_0}$
U_3	$X, Y, A, B, C \neq 0$	$\frac{U}{A} - \frac{u_0}{a_0}, \frac{Z}{A} - \frac{z_0}{a_0}$

For each $1 \leq i \leq 3$ not only U_i contains only regular points but also τU_i and $\tau^2 U_i$. This gives 9 open sets of $\Upsilon \otimes \overline{\mathbb{Q}}$ in which all points are regular. We define R_1 and R_2 to be the points $R_1 := [1 : 0 : 0 : 0 : 1 : 1 : 1]$ and $R_2 := [1 : i : 0 : i : 1 : 0 : 0]$ on $\Upsilon \otimes \overline{\mathbb{Q}}$ with $i^2 = -1$. These points are both singular. For $i = 1, 2$ let $\mathfrak{m}_i$ be the maximal ideal of the local ring $\mathcal{O}_{R_i,\Upsilon\otimes\overline{\mathbb{Q}}}$. Then the vector space $\mathfrak{m}_1/\mathfrak{m}_1^2$ is generated by the residues of $\frac{B}{A}$, $\frac{C}{A}$ and $\frac{X}{A}$, while the vector space $\mathfrak{m}_2/\mathfrak{m}_2^2$ is generated by the residues of $\frac{C}{A}$, $\frac{Z}{A}$ and $\frac{U}{A}$.

Proposition 3.2.2 *Let P be a point on $\Upsilon \otimes \overline{\mathbb{Q}}$ and suppose that for $1 \leq j, k \leq 3$ we have $P \notin \tau^k U_j \otimes \overline{\mathbb{Q}}$. Then there is an automorphism $g \in G_0$ such that $gP = R_1$ or $gP = R_2$.*

Proof. Let $P = [a : b : c : x : y : z : u]$ be an arbitrary point on Υ.

First suppose $abc \neq 0$. If $y = z = 0$, then $-a^2 = b^2 = c^2 \neq 0$ whence $x^2 = 2b^2 \neq 0$ and $u^2 = b^2 \neq 0$, so we see $P \in U_2$. Similarly, if $z = x = 0$ or $x = y = 0$, then we get $P \in \tau U_2$ or $P \in \tau^2 U_2$ respectively. Since $P \notin \tau^k U_2$ for any k this gives a contradiction, so at most one of the x, y, z equals 0 and we get $P \in \tau^k U_3$ for some k. Again a contradiction, so $abc = 0$.

Now suppose that exactly one of the coordinates a, b, c equals 0, then after applying some power of τ we may assume $ab \neq 0$ and $c = 0$. We then get $xy = \pm ab \neq 0$ and $z = \pm u$. If $z = \pm u \neq 0$, then P is contained in U_1, contradiction, so we have $z = u = 0$ and $a = \pm ib$. Now P is up to some product of the $\iota_A, \iota_B, \iota_X, \iota_Y$ equal to R_2.

Finally suppose that at least two of the coordinates a, b and c equal 0. Since $a = b = c = 0$ implies $x = y = z = u = 0$, a contradiction for homogeneous coordinates, we conclude that exactly two of the coordinates equal zero and after applying a power of τ we may assume $b = c = 0$, whence $x = 0$. Then up to some product of the $\iota_A, \iota_Y, \iota_Z, \iota_U$ the point P equals R_1. □

Corollary 3.2.3 *The 9 open subsets $\tau^k U_j \otimes \overline{\mathbb{Q}}$ with $1 \leq j, k \leq 3$ cover the regular locus of $\Upsilon \otimes \overline{\mathbb{Q}}$. The surface $\Upsilon \otimes \overline{\mathbb{Q}}$ contains exactly 48 singular points, all with a stabilizer in G_0 of order 16.*

Proof. Let P be a point not contained in the $\tau^k U_j$ for $1 \leq j, k \leq 3$. Then by Proposition 3.2.2 there is a $g \in G_0$ with $g(P) = R_1$ or $g(P) = R_2$. Since R_1 and R_2 are singular, so is P. Hence the $\tau^k U_j$ do indeed cover the regular locus of $\Upsilon \otimes \overline{\mathbb{Q}}$. The orbits of R_1 and R_2 under the action of G_0 are easily determined to be

$$\left\{\begin{array}{l}[1:0:0:0:\pm1:\pm1:\pm1],\\ [0:1:0:\pm1:0:\pm1:\pm1],\\ [0:0:1:\pm1:\pm1:0:\pm1]\end{array}\right. \qquad \text{and} \qquad \left\{\begin{array}{l}[1:\pm i:0:\pm i:\pm1:0:0],\\ [0:1:\pm i:0:\pm i:\pm1:0],\\ [\pm i:0:1:\pm1:0:\pm i:0]\end{array}\right.$$

respectively. Since the orbits of R_1 and R_2 under G_0 are different and both contain 24 different points, this implies that we get 48 singular points, all with a stabilizer in G_0 of order $384/24 = 16$. Note for instance that $\sigma R_2 = R_2$. □

Actually, there are more automorphisms of $\Upsilon \otimes \overline{\mathbb{Q}}$, but these may not be $\mathbb{Q}$-rational. With $i^2 = -1$, it is easily checked that the map $\rho\colon [a:b:c:x:y:z:u] \mapsto [a:b:iu:iy:ix:z:ic]$ gives an automorphism of $\Upsilon \otimes \overline{\mathbb{Q}}$ and that we have the following relations.

$$\begin{array}{lll}\rho^2 = \iota_A\iota_B\iota_Z, & & \\ \iota_C\rho = \rho\iota_U, & \iota_U\rho = \rho\iota_C, & \\ \iota_Y\rho = \rho\iota_X, & \iota_X\rho = \rho\iota_Y, & \\ \iota_A\rho = \rho\iota_A, & \iota_B\rho = \rho\iota_B, & \iota_Z\rho = \rho\iota_Z, \\ \sigma\rho\sigma = \rho, & \rho\tau\rho = \iota_C\iota_Y\tau\rho\tau^{-1}\sigma. & \end{array} \tag{23}$$

Let G be the group $G = G_0 \cdot \langle\rho\rangle$, where the product taken in $\mathrm{Aut}(\Upsilon \otimes \overline{\mathbb{Q}})$. Using the relations in (23) we can write any element of G as either

$$\iota_A^{q_A}\iota_B^{q_B}\iota_C^{q_C}\iota_X^{q_X}\iota_Y^{q_Y}\iota_Z^{q_Z}\tau^t\sigma^s \qquad \text{with} \quad 0 \le q_A, q_B, q_C, q_X, q_Y, q_Z, s \le 1, \quad 0 \le t \le 2. \tag{24}$$

or

$$\iota_A^{q_A}\iota_B^{q_B}\iota_C^{q_C}\iota_X^{q_X}\iota_Y^{q_Y}\iota_Z^{q_Z}\tau^{t_1}\rho\tau^{t_2}\sigma^s \qquad \text{with} \quad 0 \le q_A, q_B, q_C, q_X, q_Y, q_Z, s \le 1, \quad 0 \le t_1, t_2 \le 2. \tag{25}$$

It is also easily checked that the elements in (24) and (25) all act differently on the set of singular points of $\Upsilon \otimes \overline{\mathbb{Q}}$. This implies that G acts faithfully on the singular points and that $\#G = 2^6 \cdot 3 \cdot 2 + 2^6 \cdot 3^2 \cdot 2 = 2^9 \cdot 3 = 1536$. The group G also acts transitively on the set of singular points for we have $\rho\tau^2 R_1 = R_2$. Hence the stabilizer of each singular point has order $1536/48 = 32$.

We number the 48 isolated singularities $R_1, R_2, \ldots, R_{48}$, where R_1 and R_2 are as before. They will turn out to be *ordinary double points*. We will see that this particular type of singularity is in some way not too bad, but before we can define the notion *ordinary double point* we need another definition, see [47, I.5, p.34].

Definition 3.2.4 (analytically isomorphic) *Let X and Y be two varieties over an algebraically closed field K and P and Q points on X and Y respectively with local rings $\mathcal{O}_P$ and $\mathcal{O}_Q$. Let $\hat{\mathcal{O}}_P$ and $\hat{\mathcal{O}}_Q$ denote the completions of $\mathcal{O}_P$ and $\mathcal{O}_Q$ at their maximal ideals. Then P and Q are said to be analytically isomorphic if there is an isomorphism $\hat{\mathcal{O}}_P \cong \hat{\mathcal{O}}_Q$ as K-algebras.*

One important fact about this notion is Corollary 3.2.6 to the following lemma.

Lemma 3.2.5 *Let A be a noetherian ring and $\mathfrak{m}$ a maximal ideal. Consider the scheme $X = \operatorname{Spec} A$ with the point P corresponding to the ideal $\mathfrak{m}$. Let $\pi\colon \widetilde{X} \to X$ be the blow-up of X in the point P and let $E = \pi^{-1}(P)$ be the exceptional fibre above P. Then on $\widetilde{X}$ we have $E^2 = \deg \mathcal{O}_E(-1)$.*

Further let $A_\mathfrak{m}$ be the localization of A at $\mathfrak{m}$ and let $\hat{A}_\mathfrak{m}$ be its $\mathfrak{m}$-adic completion with maximal ideal $\hat{\mathfrak{m}} = \mathfrak{m}\hat{A}_\mathfrak{m}$. Let P' be the point on the scheme $X' = \operatorname{Spec} \hat{A}_\mathfrak{m}$ corresponding to $\hat{\mathfrak{m}}$ and let $\pi'\colon \widetilde{X}' \to X'$ be the blow-up of X' in the point P' with exceptional fibre $E' = \pi'^{-1}(P')$. Then E and E' are isomorphic and $E^2 = E'^2$.

Proof. Let P'' be the point on $X'' = \operatorname{Spec} A_{\mathfrak{m}}$ corresponding with the maximal ideal $\mathfrak{m}'' = \mathfrak{m}A_{\mathfrak{m}}$ and let $\pi'' \colon \widetilde{X}'' \to \operatorname{Spec} A_{\mathfrak{m}}$ be the blow-up of X'' at P'', write $E'' = \pi''^{-1}(P'')$. Define the graded A-algebra

$$R = \bigoplus_{k \geq 0} \mathfrak{m}^k,$$

where we take $\mathfrak{m}^0 = A$ and the graded A-algebra structure is as follows. For $f \in \mathfrak{m}^k$ and $g \in \mathfrak{m}^l$ the product fg is taken to be in $\mathfrak{m}^{k+l}$. We have $1_R = (1, 0, 0, \ldots)$. Define similarly the graded $A_{\mathfrak{m}}$-algebra R'' and the graded $\hat{A}_{\mathfrak{m}}$-algebra R' by

$$R'' = \bigoplus_{k \geq 0} \mathfrak{m}''^k, \qquad R' = \bigoplus_{k \geq 0} \hat{\mathfrak{m}}^k.$$

Since A is noetherian, we find from Theorem 7.2 in [42] that there are natural isomorphisms

$$\mathfrak{m}^k/\mathfrak{m}^{k+1} \cong \mathfrak{m}''^k/\mathfrak{m}''^{k+1} \cong \hat{\mathfrak{m}}^k/\hat{\mathfrak{m}}^{k+1} \tag{26}$$

for every $k \geq 0$. The graded ring $R/\mathfrak{m}R$ can be given by

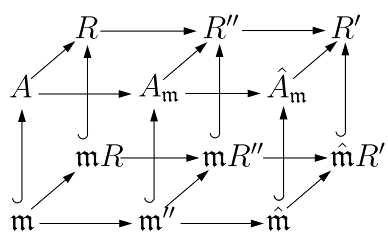
$$R/\mathfrak{m}R \cong \bigoplus_{k \geq 0} \mathfrak{m}^k/\mathfrak{m}^{k+1}, \tag{27}$$

where the degree k_0 part is given by the term with $k = k_0$. For the graded rings $R''/\mathfrak{m}R''$ and $R'/\hat{\mathfrak{m}}R'$ we find expressions similar to the one in (27), so from (26) we find the isomorphisms

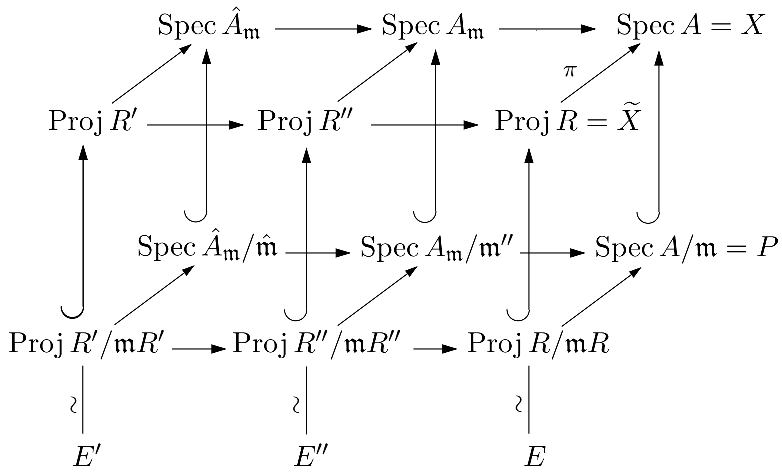
$$R/\mathfrak{m}R \cong R''/\mathfrak{m}R'' \cong R'/\hat{\mathfrak{m}}R'. \tag{28}$$

Now by definition we get $\widetilde{X} \cong \operatorname{Proj} R$ and $\widetilde{X}' \cong \operatorname{Proj} R'$ and $\widetilde{X}'' \cong \operatorname{Proj} R''$, see [47, II.7, p.163].

The morphism π is induced by the ringhomomorphism $\varphi \colon A \to R \colon a \mapsto a \cdot 1_R$ that makes R into an A-algebra. Therefore, as P corresponds to the maximal ideal $\mathfrak{m}$, the exceptional fibre $E = \pi^{-1}(P)$ can be given by $E \cong \operatorname{Proj} R/\mathfrak{m}R$. Similarly we find $E' \cong \operatorname{Proj} R'/\hat{\mathfrak{m}}R'$ and $E'' \cong \operatorname{Proj} R''/\mathfrak{m}R''$, so from the isomorphisms (28) we find that there are isomorphisms $E \cong E'' \cong E'$. This means that the commutative diagram of rings and ideals



induces the following commutative diagram of schemes in which the two lowest horizontal arrows are actually isomorphisms.



From [47, exa.V.1.4.1.] we know that the self intersection number E^2 on $\widetilde{X}$ equals

$$E^2 = \deg \mathcal{N}_{E/\widetilde{X}} = \deg_E \mathcal{H}\mathrm{om}_{\mathcal{O}_E}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_E), \tag{29}$$

where $\mathcal{I}$ is the sheaf of ideals of E in $\widetilde{X}$, i.e., $\mathcal{I}$ is the sheaf of ideals on $\widetilde{X} = \operatorname{Proj} R$ corresponding with the ideal $\mathfrak{m}R \subset R$. Writing S for the graded ring $S = R/\mathfrak{m}R$, we find from $E \cong \operatorname{Proj} R/\mathfrak{m}R$ that the sheaf $\mathcal{I}/\mathcal{I}^2$ on E is isomorphic to the sheaf of modules induced by the graded S-module $M = \mathfrak{m}R/(\mathfrak{m}R)^2$. This sheaf is written as $\widetilde{M}$, see [47, Ch.II.5, p.116–117]. From (27) and the isomorphism

$$M \cong \bigoplus_{k \geq 0} \mathfrak{m}^{k+1}/\mathfrak{m}^{k+2}$$

we see that M is isomorphic to $S(1)$, the twisted module obtained by shifting the degree of S, i.e., we have $M_k \cong S_{k+1}$ where S_k and M_k denote the degree k parts of the graded ring S and the graded S-module M respectively. By definition the sheaf on $E = \operatorname{Proj} S$ induced by the graded S-module $S(1)$ is the twisting sheaf $\mathcal{O}_E(1)$ of Serre, see [47, Ch.II.5, p.117]. Hence we get

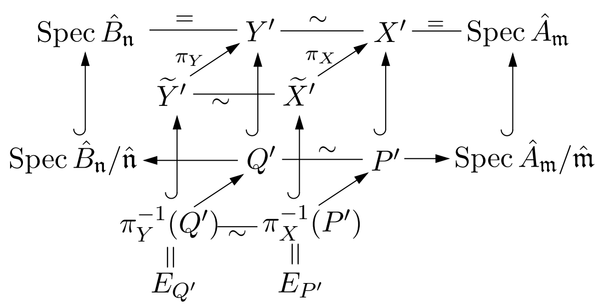
$$E^2 = \deg \mathcal{H}\mathrm{om}_{\mathcal{O}_E}(\mathcal{O}_E(1), \mathcal{O}_E) \cong \deg \mathcal{O}_E(-1).$$

Similarly we find $E'^2 = \deg \mathcal{O}_{E'}(-1)$, so from the isomorphism $E \cong E'$ we also find $E^2 = E'^2$. □

Corollary 3.2.6 *Let X and Y be varieties containing the analytically isomorphic points P and Q respectively. Let E_P and E_Q be the exceptional curves above P and Q after blowing up X and Y in P and Q respectively. Then E_P and E_Q are isomorphic and they have the same self-intersection number.*

Proof. Let $\operatorname{Spec} A$ and $\operatorname{Spec} B$ be affine neighborhoods of P and Q respectively and let $\mathfrak{m}$ and $\mathfrak{n}$ be the maximal ideals corresponding with the points P and Q. Let $A_{\mathfrak{m}}$ and $B_{\mathfrak{n}}$ be the localizations of A and B at $\mathfrak{m}$ and $\mathfrak{n}$ respectively and let $\hat{A}_{\mathfrak{m}}$ and $\hat{B}_{\mathfrak{n}}$ be their completions with maximal ideals $\hat{m}$ and $\hat{\mathfrak{n}}$ respectively. Let P' be the point on $X' = \operatorname{Spec} \hat{A}_{\mathfrak{m}}$ corresponding to the ideal $\hat{m}$ and let Q' be the point on $Y' = \operatorname{Spec} \hat{B}_{\mathfrak{n}}$ corresponding to the ideal $\mathfrak{n}$. Let $\widetilde{X}'$ be the blow-up of X' in P' with exceptional fibre $E_{P'}$ above P' and let $\widetilde{Y}'$ be the blow-up of Y' in Q' with the exceptional fibre $E_{Q'}$ above Q'. From Lemma 3.2.5 we find that E_P and $E_{P'}$ are isomorphic and that the self intersection numbers E_P^2 and $E_{P'}^2$ on $\operatorname{Spec} A \subset X$ and X' respectively are equal. Similarly we find $E_Q \cong E_{Q'}$ and $E_Q^2 = E_{Q'}^2$, so it suffices to show that $E_{P'}$ and $E_{Q'}$ are isomorphic and that $E_{P'}^2 = E_{Q'}^2$.

The fact that P and Q are analytically isomorphic says that there exists an isomorphism $\hat{A}_{\mathfrak{m}} \to \hat{B}_{\mathfrak{n}}$ mapping $\hat{\mathfrak{m}}$ to $\hat{\mathfrak{n}}$. Then clearly we have $E_{P'} \cong E_{Q'}$ and $E_{P'}^2 = E_{Q'}^2$.



□

Definition 3.2.7 *Let X be an algebraic surface and P a point on X. Let $V = (0,0,0)$ be the vertex of the cone in $\mathbb{A}^3$ given by $x^2 + y^2 + z^2 = 0$. Then we call P an* ordinary double point *or* conical double point *if P is analytically isomorphic to V.*

The fact that ordinary double points are in some way not too bad is made precise in the following lemma. The remark on the self-intersection number will be needed later.

Lemma 3.2.8 *Let X be an algebraic surface over an algebraically closed field K and let P on X be analytically isomorphic to the vertex of the cone over a nonsingular curve $C \subset \mathbb{P}_K^2$ of degree d. Let $\varphi\colon \widetilde{X} \to X$ be the blowing-up of X at P and let $E = \varphi^{-1}(P)$ be the exceptional curve above P. Then $\widetilde{X}$ is nonsingular in an open neighborhood of the curve E, which is isomorphic to C and $E^2 = -d$. In particular, if P is an ordinary double point, then E is isomorphic to $\mathbb{P}^1$ and $E^2 = -2$.*

Proof. By Lemma 3.2.6 the fibre above P of the blow-up in P depends only on the completion of the local ring at P, so we only need to prove this in the case that P is the origin $(0,0,0) \in \mathbb{A}^3$ and X is the surface in $\mathbb{A}^3$ determined by $F = 0$, where $F \in K[X,Y,Z]$ is a homogeneous polynomial describing C. In that case the blow-up of X is one of the standard examples of blowing up. It follows for instance from [47, exc.I.5.7.] that E is isomorphic to the curve C and that $\widetilde{X}$ is nonsingular in an open neigboorhood of E. This also follows from [43, Exa.IV.27] in the case $F = X^2 + Y^2 - Z^2$. From Lemma 3.2.5 we find $E^2 = \deg_E \mathcal{O}_E(-1)$, so from the fact that $E \cong C$ we find that $E^2 = \deg_C \mathcal{O}_C(-1) = -\deg_C \mathcal{O}_C(1)$. Let the embedding of C in $\mathbb{P}^2$ be given by $j\colon C \hookrightarrow \mathbb{P}^2$. Then we have $\mathcal{O}_C(1) \cong j^*\mathcal{O}_{\mathbb{P}}(1)$ and if H is a line in $\mathbb{P}^2$ and D is the divisor $D = H \cap C$ on C then $j^*\mathcal{O}(1) \cong j^*\mathcal{L}(H) \cong \mathcal{L}(D)$. Since $\deg_C \mathcal{L}(D) = \deg_C D$ (see [47, exc.II.6.12]) and $\deg_C D = (\deg C)(\deg H) = \deg C = d$ (Theorem of Bézout), we find

$$E^2 = -\deg_C \mathcal{O}_C(1) = -\deg_C j^*\mathcal{O}_{\mathbb{P}}(1) = -\deg_C \mathcal{L}(D) = -\deg_C D = -d.$$

This also follows from example V.2.11.4 on page 374 of [47] for $n = 2$. If P is an ordinary double point, then we can take C to be the curve of degree $d = 2$ given by $X^2 + Y^2 + Z^2 = 0$. □

Lemma 3.2.9 *All singular points on $\Upsilon \otimes \overline{\mathbb{Q}}$ are ordinary double points.*

Proof. Since $G \subset \mathrm{Aut}(\Upsilon \otimes \overline{\mathbb{Q}})$ acts transitively on the 48 singular points, we only need to show that $R_1 = [1:0:0:0:1:1:1]$ is an ordinary double point. Since R_1 lies in the affine part $A \neq 0$, the local ring $\mathcal{O}_{R_1, \Upsilon\otimes\overline{\mathbb{Q}}}$ can be given by

$$(\overline{\mathbb{Q}}[b,c,x,y,z,u]/I)_{\mathfrak{m}},$$

with

$$
\begin{aligned}
I &= (\quad b^2+c^2-x^2, \quad (1-z)(1+z)-b^2, \quad (1-y)(1+y)-c^2, \quad (1-u)(1+u)-x^2 \quad),\\
\mathfrak{m} &= (b,c,x,y-1,z-1,u-1).
\end{aligned} \tag{30}
$$

There are $f_y, f_z, f_u \in \hat{\mathcal{O}}_{R_1,\Upsilon\otimes\overline{\mathbb{Q}}}$ such that $f_y^2 = 1+y$, $f_z^2 = 1+z$ and $f_u^2 = 1+u$, namely

$$
f_y = \sqrt{2} + \frac{1}{2}\sqrt{2}(y-1) - \frac{1}{4}\sqrt{2}(y-1)^2 + \dots
$$

and similar f_z and f_u. These elements are units, so we can define $b' = bf_z^{-1}$, $c' = cf_y^{-1}$ and $x' = xf_u^{-1}$ to find

$$
\begin{aligned}
\hat{\mathcal{O}}_{R_1,\Upsilon\otimes\overline{\mathbb{Q}}} &\cong \left(\overline{\mathbb{Q}}[b',c',x',y,z,u]/(b'^2+c'^2-x'^2, z-1-b'^2, y-1-c'^2, u-1-x'^2)\right)_{\mathfrak{m}}\\
&\cong \left(\overline{\mathbb{Q}}[b',c',x']/(b'^2+c'^2-x'^2)\right)_{(b',c',x')}.
\end{aligned}
$$

By substituting $x' = ix''$ this shows that R_1 is indeed an ordinary double point. □

Definition 3.2.10 *Let $\pi\colon \widetilde{\Upsilon} \to \Upsilon$ be the blowing-up of Υ in the 48 singular points and for $i = 1,\dots,48$ let E_i be the exceptional fibre of π above R_i.*

Corollary 3.2.11 *The surface $\widetilde{\Upsilon}\otimes\overline{\mathbb{Q}}$ is nonsingular. For each i the fibre E_i is isomorphic to $\mathbb{P}^1$ and has self intersection number -2.*

Proof. This follows directly from Lemma 3.2.8 and 3.2.9. □

Lemma 3.2.12 *Let A be a commutative ring, R an A-algebra, A' the integral closure of A in R ans S a multiplicative subset of A. Then the integral closure of $S^{-1}A$ in $S^{-1}R$ is $S^{-1}A'$.*

Proof. See [37, V.1.5, Prop. 16]. □

Corollary 3.2.13 *Let A be an integrally closed domain and S a multiplicative subset of A such that $0 \notin S$. Then $S^{-1}A$ is integrally closed as well.*

Proof. The field of fractions R of A is also the field of fractions of $S^{-1}A$ since $0 \notin S$. The corollary follows by applying Lemma 3.2.12 to R with $A' = A$. □

Lemma 3.2.14 *Every commutative regular local ring is factorial, whence integrally closed.*

Proof. See [42, Thm. 19.19]. □

Lemma 3.2.15 *The surface $\Upsilon\otimes\overline{\mathbb{Q}}$ is normal.*

Proof. Since regular local rings are integrally closed by Lemma 3.2.14, we only need to show that the local rings $B_j = \mathcal{O}_{R_j,\Upsilon\otimes\overline{\mathbb{Q}}}$ at the singular points R_j are integrally closed. As the group $G \subset \mathrm{Aut}(\Upsilon\otimes\overline{\mathbb{Q}})$ acts transitively on the singular points, it suffices to do this for $j=1$. Since R_1 is contained in the affine part $A \neq 0$, we can describe $B = B_1$ as

$$
B \cong \left(\overline{\mathbb{Q}}[b,c,x,y,z,u]/I\right)_{\mathfrak{m}},
$$

where I and $\mathfrak{m}$ are as in (30). By Corollary 3.2.13 it suffices to show that $A := \overline{\mathbb{Q}}[b,c,x,y,z,u]/I$ is integrally closed. The fact that this is sufficient also follows from the fact that an affine variety is normal if and only if its coordinate ring is integrally closed, see [47, exc.I.3.17.].

The embedding $\overline{\mathbb{Q}}[b,c] \hookrightarrow \overline{\mathbb{Q}}[b,c,x,y,z,u]$ induces an embedding $\overline{\mathbb{Q}}[b,c] \hookrightarrow A$ as $\overline{\mathbb{Q}}[b,c] \cap I = (0)$. Let $Q(A)$ be the quotient field of A and let $t \in Q(A)$ be an element which is integral over A. Using the fact that $x^2, y^2, z^2, u^2 \in \overline{\mathbb{Q}}[b,c]$, we can write

$$t = f + xf_x + yf_y + zf_z + uf_u,$$

with $f, f_x, f_y, f_z, f_u \in \overline{\mathbb{Q}}(b,c)$. Since $u \mapsto -u$ induces an automorphism of A, the element

$$t' = f + xf_x + yf_y + zf_z - uf_u$$

is integral over A as well, whence so is $t - t' = 2uf_u$ and since $\operatorname{char} \overline{\mathbb{Q}} \neq 2$, so is uf_u. Similarly, so are xf_x, yf_y and zf_z and hence

$$f, \quad (xf_x)^2 = (b^2+c^2)f_x^2, \quad (1+c^2)f_y^2, \quad (1+b^2)f_z^2 \quad \text{and} \quad (1+b^2+c^2)f_u^2. \tag{31}$$

Since $\overline{\mathbb{Q}}[b,c]$ is a unique factorization domain, it is integrally closed. This implies that the elements of (31), which are all contained in $\overline{\mathbb{Q}}(b,c)$ and integral over $\overline{\mathbb{Q}}[b,c]$, are actually all in $\overline{\mathbb{Q}}[b,c]$. Since the factor b^2+c^2 is squarefree, it follows from $(b^2+c^2)f_x^2 \in \overline{\mathbb{Q}}[b,c]$ that $f_x^2 \in \overline{\mathbb{Q}}[b,c]$, whence $f_x \in \overline{\mathbb{Q}}[b,c]$. Analogously, since the factors $1+b^2$, $1+c^2$ and $1+b^2+c^2$ are also squarefree, we get $f_z, f_y, f_u \in \overline{\mathbb{Q}}[b,c]$. Therefore we have $t \in A$, so A is integrally closed. □

Remark 3.2.16 It is a fact that if P is an ordinary double point of a surface S, then S is normal at P. And more generally, if P is an isolated hypersurface singularity on a surface S, i.e., P is analytically isomorphic to an isolated singular point on a hypersurface $S' \subset \mathbb{P}^3$, then S is normal at P. This follows from the following two facts. First that an irreducible affine hypersurface is normal if and only if it is regular in codimension one, see [54, Ch.III.8, Prop.2]. Secondly the more heavy machinery, stating that a local excellent ring is normal if and only if its completion is, see [46, Sch.IV.7.8.3(v)].

Since almost all surfaces that we will encounter in this paper are complete intersections, we will not prove the facts stated in Remark 3.2.16 or even refer to the complete proofs. Instead we will use the following proposition to prove that our surfaces are normal. Lemma 3.2.15 also follows from this proposition.

Proposition 3.2.17 *Let Y be a locally complete intersection subscheme of a nonsingular variety X over an algebraically closed field K. Then*

(a) Y is Cohen-Macaulay,

(b) Y is normal if and only if it is regular in codimension 1*.*

Proof. See [47, Prop.II.8.23]. □

Lemma 3.2.18 *The Hilbert polynomial P_Υ of the surface Υ is given by $P_\Upsilon(n) = 8(n^2-n+1)$ and the arithmetic genus $p_a(\Upsilon)$ of Υ equals* 7*. The Euler characteristic $\chi(\Upsilon)$ equals* 8*.*

Proof. The degree n part S_n of the graded homogeneous coordinate ring $S = S(\Upsilon)$ is generated by the residues of degree n monomials $A^{q_a}B^{q_b}C^{q_c}X^{q_X}Y^{q_Y}Z^{q_Z}U^{q_U}$. Since we can express X^2, Y^2, Z^2 and U^2 in A, B and C, a basis for S_n over $\mathbb{Q}$ is given by those monomials with $q_X, q_Y, q_Z, q_U \in \{0,1\}$. Consider these monomials. The number of 3-tuples (q_A, q_B, q_C) with $q_A+q_B+q_C = n-k$ is $\binom{n-k+2}{2}$, so for $0 \le k \le 4$ there are exactly $\binom{n-k+2}{2}$ of these monomials with k of the q_X, q_Y, q_Z, q_U fixed to be equal to 1 and the other $4-k$ equal to 0. For $n \ge 4$ this gives $\sum_{k=0}^4 \binom{4}{k}\binom{n-k+2}{2} = 8(n^2-n+1)$ monomials that form a basis of S_n over K. From Lemma 3.1.60 and Definition 3.1.51 we find that $\chi(\Upsilon) = P_\Upsilon(0) = 8$ and the arithmetic genus $p_a(\Upsilon)$ equals $(-1)^{\dim \Upsilon}(P_\Upsilon(0)-1) = 7$, see [47, exc.I.7.2]. □

Since the arithmetic genus is invariant under monoidal transformations, the arithmetic genus $p_a(\widetilde{\Upsilon})$ of $\widetilde{\Upsilon}$ equals 7 as well and we also have $\chi(\widetilde{\Upsilon}) = 8$. We will see that the geometric genus $p_g(\widetilde{\Upsilon})$ coincides with the arithmetic genus. Actually we shall now compute all Hodge numbers of $\widetilde{\Upsilon}$ but $h^{1,1}$ which we will compute later, see Corollary 3.3.34.

Proposition 3.2.19 *Set $q = h^{1,0}(\widetilde{\Upsilon} \otimes \mathbb{C})$, then the Hodge numbers of $\widetilde{\Upsilon} \otimes \mathbb{C}$ are given by*

$$\begin{array}{|c|c|c|}\hline h^{0,2} & h^{1,2} & h^{2,2} \\ \hline h^{0,1} & h^{1,1} & h^{2,1} \\ \hline h^{0,0} & h^{1,0} & h^{2,0} \\ \hline \end{array} \quad = \quad \begin{array}{|c|c|c|}\hline 7+q & q & 1 \\ \hline q & h^{1,1} & q \\ \hline 1 & q & 7+q \\ \hline \end{array}.$$

Hence we have $p_g(\widetilde{\Upsilon}) = 7 + q$.

Proof. Unless stated otherwise the Hodge and betti numbers $h^{p,q}$ and b_k denote those of $\widetilde{\Upsilon} \otimes \mathbb{C}$. Since $\widetilde{\Upsilon} \otimes \mathbb{C}$ is projective we have $H^0(\widetilde{\Upsilon} \otimes \mathbb{C}, \mathcal{O}_{\widetilde{\Upsilon} \otimes \mathbb{C}}) = \mathbb{C}$, so $b_0 = h^{0,0} = 1$ and from Poincaré duality 3.1.48 we also find $h^{2,2} = b_4 = 1$. From Proposition 3.1.47 we find that $h^{0,1} = h^{1,0} = q$, so $b_1 = h^{0,1} + h^{1,0} = 2q$. The Poincaré duality then implies that $b_3 = b_1 = 2q$ and as we have $b_3 = h^{2,1} + h^{1,2}$ and $h^{1,2} = h^{2,1}$, we find $h^{2,1} = h^{1,2} = q$.

For arithmetic genus $p_a = p_a(\widetilde{\Upsilon} \otimes \mathbb{C}) = p_a(\widetilde{\Upsilon}) = p_a(\Upsilon) = 7$ we have $7 = p_a = p_g - h^{2,1} = p_g - q = h^{2,0} - q$ from Remark 3.1.57. Therefore we get $h^{2,0} = p_g = 7+q$. Finally from Proposition 3.1.47 we also get $h^{0,2} = h^{2,0} = 7 + q$. □

Bluff 1 *we have $H^1(\widetilde{\Upsilon}, \mathcal{O}_{\widetilde{\Upsilon}}) = 0$, whence $h^{1,0}(\widetilde{\Upsilon}) = 0$ and $p_g(\widetilde{\Upsilon}) = 7$ and*

$$\begin{array}{|c|c|c|}\hline h^{0,2} & h^{1,2} & h^{2,2} \\ \hline h^{0,1} & h^{1,1} & h^{2,1} \\ \hline h^{0,0} & h^{1,0} & h^{2,0} \\ \hline \end{array} \quad = \quad \begin{array}{|c|c|c|}\hline 7 & 0 & 1 \\ \hline 0 & h^{1,1} & 0 \\ \hline 1 & 0 & 7 \\ \hline \end{array}.$$

Proof. There is no proof yet, for otherwise this result wouldn't be a bluff. □

3.3 Computing canonical sheaves and Kodaira dimensions

There are several ways to classify surfaces, one of which is by the notion of Kodaira dimension, which we will define in this section. The Kodaira dimension of a variety X is a birational invariant integer $-1 \leq \kappa(X) \leq \dim X$. If $\kappa(X) = \dim X$ then we say that X is of general type. Our main goal is to prove that the surface Υ in $\mathbb{P}^6$ describing perfect cuboids is of general type, but we will state more general propositions. In this section K will always denote an algebraically closed field of characteristic $\neq 2$. Also in this section, by abuse of notation, Υ and $\widetilde{\Upsilon}$ will denote the varieties $\Upsilon \otimes \overline{\mathbb{Q}}$ and $\widetilde{\Upsilon} \otimes \overline{\mathbb{Q}}$ over $\overline{\mathbb{Q}}$ respectively.

Let X be a projective variety over K and let $\mathcal{L}$ be an invertible sheaf on X. Consider the K-vectorspace $H^0(X, \mathcal{L}) = \Gamma(X, \mathcal{L})$ of global sections. Following K. Ueno [60, II.5], we define

$$\mathbb{N}(\mathcal{L}, X) := \{m \in \mathbb{Z}_{>0} : \dim_K H^0(X, \mathcal{L}^m) \neq 0\}.$$

Actually Ueno only does this for invertible sheaves associated to divisors. Note that $\mathbb{N}(\mathcal{L}, X)$ is a semi-group under addition, for if we have $f \in H^0(X, \mathcal{L}^m)$ and $g \in H^0(X, \mathcal{L}^n)$, then we also have $f \otimes g \in H^0(X, \mathcal{L}^{m+n})$. Since X is projective, the dimension of $H^0(X, \mathcal{L})$ over K is finite. Hence for $m \in \mathbb{N}(\mathcal{L}, X)$ we can choose a basis $\phi_0, \ldots, \phi_N$ for $H^0(X, \mathcal{L}^m)$.

Let P be a point on X. As $\mathcal{L}$ is an invertible sheaf we can choose an open neighborhood U of P such that $\mathcal{L}(U)$ is a free $\mathcal{O}_X(U)$-module of rank 1, say generated by $s \in \mathcal{L}(U)$. Hence there are regular functions $g_0, \ldots, g_N \in \mathcal{O}_X(U)$ such that $\phi_j|_U = g_j \cdot s$. Note that s is not a unique generator of $\mathcal{L}(U)$ as a free $\mathcal{O}_X(U)$-module, whence neither are the g_j. However the ratios $g_i : g_j$ are unique, so we can define a rational map

$$f_m: X \dashrightarrow \mathbb{P}^N_K: P \mapsto [g_0(P) : g_1(P) : \cdots : g_N(P)],$$

which we will also write as

$$f_m: X \dashrightarrow \mathbb{P}^N_K: P \mapsto [\phi_0(P) : \phi_1(P) : \cdots : \phi_N(P)],$$

without addressing the g_j. If $\mathcal{L}^m$ is generated by its global sections, then f_m is a morphism. In general it determines a morphism from a maximal open set $U_m \subset X$ to projective space. The set $X \setminus U_m$ is called the set of basepoints. For a different basis of $H^0(X, \mathcal{L}^m)$ we get a rational map $f'_m: X \dashrightarrow \mathbb{P}^N$ which differs from f_m only by the action of an element of $\mathrm{GL}(N+1, K)$ on $\mathbb{P}^N_K$. Hence $f_m(U_m)$ and $f'_m(U_m)$ are isomorphic and their dimensions are equal. Therefore the following definition is well defined.

Definition 3.3.1 *Let X be a projective variety over K and let $\mathcal{L}$ be an invertible sheaf on X. Then we define the* invertible sheaf dimension *$\kappa(\mathcal{L}, X)$ of $\mathcal{L}$ on X by*

$$\kappa(\mathcal{L}, X) = \begin{cases} -1 & \text{if} \quad \mathbb{N}(\mathcal{L}, X) = \emptyset, \\ \max_{m \in \mathbb{N}(\mathcal{L}, X)}(\dim f_m(U_m)) & \text{if} \quad \mathbb{N}(\mathcal{L}, X) \neq \emptyset. \end{cases}$$

Lemma 3.3.2 *Let X be a projective variety over K and let $\mathcal{L}$ and $\mathcal{L}'$ be two isomorphic invertible sheaves on X. Then we have $\kappa(\mathcal{L}, X) = \kappa(\mathcal{L}', X)$.*

Proof. Let $\psi: \mathcal{L} \to \mathcal{L}'$ be an isomorphism and let $\phi_0, \ldots, \phi_N$ be a basis of $H^0(X, \mathcal{L}^m)$ giving a rational map $f_m: X \dashrightarrow \mathbb{P}^N$. Then the $\psi\phi_j$ form a basis of $H^0(X, \mathcal{L}'^m)$ giving a rational map $f'_m: X \dashrightarrow \mathbb{P}^N$. Let P be a point on X and let $U \subset X$ be an open neighborhood of P such that $\mathcal{L}(U)$ and $\mathcal{L}'(U)$ are free $\mathcal{O}_X(U)$-modules of rank 1. Let $s \in \mathcal{L}(U)$ be a generator of $\mathcal{L}(U)$ as an $\mathcal{O}_X(U)$-module. Then $\psi(s)$ is a generator of $\mathcal{L}'(U)$ and if $g_j \in \mathcal{O}_X(U)$ satisfies $\phi_j|_U = g_j \cdot s$, then $\psi\phi_j|_U = g_j \cdot \psi(s)$, so both f_m and f'_m can be given by

$$X \dashrightarrow \mathbb{P}^N_K: P \mapsto [g_0(P) : g_1(P) : \cdots : g_N(P)].$$

□

From this lemma it follows that we may define the divisor dimension of a Cartier divisor as follows.

Definition 3.3.3 *Let X be a projective variety over K and let D be a Cartier divisor on X. Then the* divisor dimension *of D is defined by $\kappa(D, X) := \kappa(\mathcal{L}(D), X)$, where $\mathcal{L}(D)$ is the invertible sheaf associated to D.*

From Definition 3.3.1 it is clear that $-1 \leq \kappa(\mathcal{L}, X) \leq \dim X$ for any invertible sheaf on a projective variety over K.

Definition 3.3.4 *We say that an invertible sheaf $\mathcal{L}$ on a projective variety X over K is pseudo-ample if $\kappa(\mathcal{L}, X) = \dim X$. We say that a Cartier divisor is pseudo-ample if $\kappa(D, X) = \dim X$.*

Remark 3.3.5 Let X again be a projective variety over K and let $\mathcal{L}$ be an invertible sheaf on X. Suppose $m \in \mathbb{N}(\mathcal{L}, X)$, let $\phi_0, \ldots, \phi_N$ be a basis for $H^0(X, \mathcal{L}^m)$ and let $f_m: X \dashrightarrow \mathbb{P}^N$ be the corresponding rational map. Let $U \subset X$ be an open subset such that $\mathcal{L}(U)$ is a free $\mathcal{O}_X(U)$-module and let $U_m \subset X$ be the maximal open subset on which f_m is defined. Let W_m be defined by $W_m = f_m(U_m)$, then $\dim W_m = \dim f_m(U_m) = \dim f_m(U)$. Let $s \in \mathcal{L}(U)$ and $g_0, \ldots, g_N \in \mathcal{O}_X(U)$ be such that $\phi_j|_U = g_j \cdot s$. Then on U the rational map f_m is defined by

$$U \dashrightarrow \mathbb{P}^N: P \mapsto [g_0(P) : g_1(P) : \cdots : g_N(P)].$$

Hence the function field of $f_m(U)$ is $K(\frac{g_1}{g_0}, \ldots, \frac{g_N}{g_0})$ and $\dim W_m = \dim f_m(U)$ equals the transcendence degree of this field over K.

Lemma 3.3.6 *Let X be a projective variety over K and let $\mathcal{L}$ be an invertible sheaf on X. Let $m \in \mathbb{N}(\mathcal{L}, X)$ be an integer. Let $\psi_0, \dots, \psi_r$ be elements of $H^0(X, \mathcal{L}^m)$ and let h be the rational map*

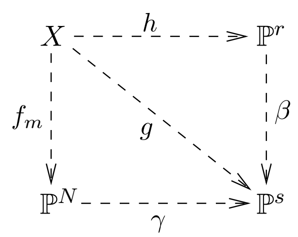
$$h: X \dashrightarrow \mathbb{P}^r: P \mapsto [\psi_0(P) : \cdots : \psi_r(P)].$$

Then $\kappa(\mathcal{L}, X) \geq \dim h(X)$.

Proof. Let $s+1$ be the dimension of $\text{span}(\psi_0, \dots, \psi_r)$. After a permutation we may assume that $\psi_0, \dots, \psi_s$ are linearly independent over K. We can extend them with elements $\phi_{s+1}, \dots, \phi_N$ to a basis of $H^0(X, \mathcal{L}^m)$. Then we have two more rational maps

$$g: X \dashrightarrow \mathbb{P}^s: P \mapsto [\psi_0(P) : \cdots : \psi_s(P)],$$
$$f_m: X \dashrightarrow \mathbb{P}^N: P \mapsto [\psi_0(P) : \cdots : \psi_s(P) : \phi_{s+1} : \cdots : \phi_N(P)].$$

Let $\beta: \mathbb{P}^r \dashrightarrow \mathbb{P}^s$ be the projection on the first $s+1$ coordinates. Then we have $g = \beta \circ h$ so clearly we have $\dim h(X) \geq \dim g(X)$. However, since the $\psi_{s+1}, \dots, \psi_r$ are K-linear combinations of the $\psi_0, \dots, \psi_s$, we also find $\dim h(X) \leq \dim g(X)$, so $\dim h(X) = \dim g(X)$. Let $\gamma: \mathbb{P}^N \dashrightarrow \mathbb{P}^s$ be the projection on the first $s+1$ coordinates. Then we have $\dim f_m(X) \geq \dim g(X) = \dim h(X)$, whence $\kappa(\mathcal{L}, X) \geq \dim h(X)$.



□

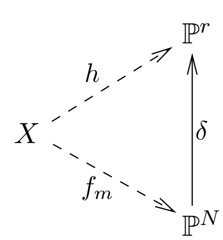
Corollary 3.3.7 *Let X be a projective variety over K and let $\mathcal{L}$ be an invertible sheaf on X. Then we have $\kappa(\mathcal{L}, X) = \kappa(\mathcal{L}^k, X)$ for all positive integers k.*

Proof. Clearly we have $k \cdot \mathbb{N}(\mathcal{L}^k, X) \subset \mathbb{N}(\mathcal{L}, X)$, whence $\kappa(\mathcal{L}^k, X) \leq \kappa(\mathcal{L}, X)$, as for the latter we are taking the maximum of a larger set of integers. It remains to show the converse. If $\kappa(\mathcal{L}, X) = -1$, the converse is evidently true, so we may assume that $\mathbb{N}(\mathcal{L}, X) \neq \emptyset$. Let $m \in \mathbb{N}(\mathcal{L}, X)$ be a positive integer such that for a basis $\phi_0, \dots, \phi_N$ of $H^0(X, \mathcal{L}^m)$ and the associated rational map $f_m: X \dashrightarrow \mathbb{P}^N$ we have $\dim f_m(X) = \kappa(\mathcal{L}, X)$.

Consider the $r+1 = \binom{N+k}{k}$ elements $\Phi_0, \dots, \Phi_r \in H^0(X, (\mathcal{L}^k)^m)$ of the form $\phi_{j_1} \otimes \cdots \otimes \phi_{j_k}$ with $0 \leq j_1 \leq \ldots \leq j_k \leq N$. Let h be the rational map

$$h: X \dashrightarrow \mathbb{P}^r: P \mapsto [\Phi_0(P) : \cdots : \Phi_r(P)].$$

From Lemma 3.3.6 we conclude that $\kappa(\mathcal{L}^k, X) \geq \dim h(X)$, so it suffices to prove that $\dim h(X) = \dim f_m(X)$. Let $\delta: \mathbb{P}^N \to \mathbb{P}^r$ be the k-uple embedding. Then we have $h = \delta \circ f_m$ and since δ induces an isomorphism from $\mathbb{P}^N$ onto its image (see [47, Exc.I.3.4]), we conclude that indeed $\dim h(X) = \dim f_m(X)$.



□

Lemma 3.3.8 *Let X be a normal projective variety and $\rho: X \to \mathbb{P}^n$ a morphism. Let $m > 0$ be an integer, then for the sheaf $\mathcal{L} = \rho^*\mathcal{O}(m)$ on X we get $\dim \rho(X) \leq \kappa(\mathcal{L}, X) \leq \dim X$.*

Proof. By Proposition 3.3.7 it suffices to prove this for $m = 1$, so we assume $\mathcal{L} = \rho^*\mathcal{O}(1)$. The K-vectorspace $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}}(1))$ of global sections of $\mathcal{O}_{\mathbb{P}}(1)$ is generated by the coordinates $x_0, \ldots, x_n$ of $\mathbb{P}^n$. Hence $H^0(X, \mathcal{L})$ contains the sections $\rho^*x_0, \ldots, \rho^*x_n$. Let h be the rational map

$$h: X \dashrightarrow \mathbb{P}^n: P \mapsto [\rho^*x_0(P) : \cdots : \rho^*x_n(P)].$$

Then from Lemma 3.3.6 we conclude that $\kappa(\mathcal{L}, X) \geq \dim h(X)$, but h is nothing but the morphism ρ, whence $\kappa(\mathcal{L}, X) \geq \dim \rho(X)$, which proves the left inequality. The right inequality is a general fact that we have seen before. $\square$

In fact, if X is a projective variety and $\rho: X \to \mathbb{P}^n$ is morphism such that $\dim \rho(X) = \dim X$, then we can say even more, for which we will first prove the following lemma.

Lemma 3.3.9 *Let X be an integral projective scheme of dimension ≥ 1 over K, and let $\mathcal{L}$ be a pseudo-ample invertible sheaf on X. Then $H^0(X, \mathcal{L}^{-1}) = 0$ and $\kappa(\mathcal{L}^{-1}, X) = -1$.*

Proof. Since $\mathcal{L}$ is pseudo-ample, there is a positive integer n such that a basis of $H^0(X, \mathcal{L}^n)$ gives a rational map $f: X \to \mathbb{P}^N$ with

$$\dim X = \dim f(X) \leq N = \dim_K H^0(X, \mathcal{L}^n) - 1.$$

Therefore we get $\dim_K H^0(X, \mathcal{L}^n) \geq \dim X + 1 \geq 2$, so $\mathcal{L}^n$ admits at least two nonzero linearly independent global sections, say f_1 and f_2. Now suppose that $H^0(X, \mathcal{L}^{-1}) \neq 0$. Then $\mathcal{L}^{-1}$ admits a nonzero regular global section as well, say g, and hence so does $\mathcal{L}^{-n}$, namely g^n. Then $f_i g^n$ is a nonzero global section of $\mathcal{L}^n \otimes \mathcal{L}^{-n} \cong \mathcal{O}_X$. Since $\mathcal{O}_X$ admits only the constants as global sections, $f_i g^n$ is a nonzero constant for $i = 1, 2$, contradicting the fact that f_1 and f_2 are linearly independent. For every positive integer m the sheaf $\mathcal{L}^m$ is pseudo-ample as well, whence $H^0(X, \mathcal{L}^{-m}) = 0$ and $\mathbb{N}(\mathcal{L}^{-1}, X) = \emptyset$, so $\kappa(\mathcal{L}^{-1}, X) = -1$. $\square$

Lemma 3.3.10 *Let X be a normal projective variety and $\rho: X \to \mathbb{P}^n$ a morphism such that $\dim \rho(X) = \dim X \geq 1$. For an integer m consider the sheaf $\mathcal{L} = \rho^*\mathcal{O}(m)$ on X. Then*

$$\begin{array}{ll} \kappa(\mathcal{L}, X) = -1 & \text{if } m < 0, \\ \kappa(\mathcal{L}, X) = 0 & \text{if } m = 0, \\ \kappa(\mathcal{L}, X) = \dim X & \text{if } m > 0. \end{array}$$

Proof. If $m > 0$ then from Lemma 3.3.8 we find that $\dim X = \kappa(\mathcal{L}, X)$ and $\mathcal{L}$ is pseudo-ample. If $m = 0$, then we get $\mathcal{L} \cong \mathcal{O}_X$, which has only the constants as global sections for X is projective. Therefore the map f_m corresponding to a basis of $H^0(X, \mathcal{L}^m) \cong H^0(X, \mathcal{O}_X) \cong K$ is constant for each m, so $\kappa(\mathcal{L}, X) = 0$. If $m < 0$, then we have just proven that $\mathcal{L}^{-k}$ is pseudo-ample for every integer $k \geq 1$, whence from Lemma 3.3.9 we find that $\kappa(\mathcal{L}, X) = -1$. $\square$

Definition 3.3.11 (Kodaira dimension) *For a nonsingular projective variety X we define the Kodaira dimension to be $\kappa(X) = \kappa(\omega_X, X)$, where ω_X is the canonical sheaf. The map f_m corresponding to a basis of $H^0(X, \omega_X^m)$ will be called the* m-canonical map *of X.*

The Kodaira dimension is a birational invariant, i.e., for two birationally equivalent nonsingular projective varieties X and X' we have $\kappa(X) = \kappa(X')$, see [47, Thm. II.8.19]. In characteristic 0, given a projective variety X, there exists a birationally equivalent nonsingular projective variety $\widetilde{X}$ as resolution of singularities is known due to Hironaka. Lipman proved that for a *surface* S there exists birationally equivalent nonsingular projective surface in arbitrary characteristic. Therefore the following definition gives a well defined birational invariant.

Definition 3.3.12 *Let X be an arbitrary projective variety over K and assume either* $\operatorname{char} K = 0$ *or* $\dim X \leq 2$. *The Kodaira dimension of X is defined to be the Kodaira dimension of a birationally equivalent nonsingular projective variety $\widetilde{X}$.*

Clearly there is the inequality $-1 \leq \kappa(X) \leq \dim X$. We can classify varieties of a given dimension by the Kodaira dimension. Note that by definition we have $\kappa(X) = \dim X$ if and only if the sheaf ω_X is pseudo-ample. Some people call X in this case pseudo-canonical. We will use the following term.

Definition 3.3.13 *If X is a variety of dimension $n = \dim X \geq 1$ over a field k such that $X \otimes \overline{k}$ has Kodaira dimension $\kappa(X) = n$, then we say that X is of* general type.

If X is a nonsingular variety over K of dimension n with function field $K(X)$, and if ω_0 and ω_1 are two nonzero differentials on X of degree n, then there is a rational function $f \in K(X)$ such that $\omega_1 = f \cdot \omega_0$. Using this we define algebraic independence of differentials of degree $\dim X$.

Definition 3.3.14 *Let X be a nonsingular variety of dimension n over K with function field $K(X)$ and let $\omega_0, \omega_1, \ldots, \omega_k$ be nonzero differentials on X of degree n. Let $f_1, \ldots, f_k \in K(X)$ be such that $\omega_j = f_j \cdot \omega_0$ for all $1 \leq j \leq k$. Then $\omega_0, \ldots, \omega_k$ are said to be* algebraically independent *over K if the rational functions $f_1, \ldots, f_k$ are.*

Lemma 3.3.15 *Let X be a nonsingular variety over K of dimension n and let $\omega_0, \ldots, \omega_k$ be $k+1$ algebraically independent regular differentials of degree n. Then we have $\kappa(X) \geq k$. In particular, X is of general type if there exist $n+1$ algebraically independent regular differentials of degree n on X.*

Proof. The differentials $\omega_0, \ldots, \omega_k$ are algebraically, whence linearly independent over K, so we can extend them with $\omega_{k+1}, \ldots, \omega_r$ to a basis $\omega_0, \ldots, \omega_r$ of $H^0(X, \omega_X)$ over K. For j with $k < j \leq r$ let $f_j \in K(X)$ be such that $\omega_j = f_j \cdot \omega_0$. Then the 1-canonical map of X is given by the rational map

$$f \colon X \dashrightarrow \mathbb{P}^r : P \mapsto [1 : f_1 : \cdots : f_k : f_{k+1} : \cdots : f_r].$$

Let U be the largest open subset of X on which f is well defined and set $W := f(U)$. Then we have $\kappa(X) \geq \dim W = \operatorname{tr.deg}_K K(W)$, where $K(W)$ is the function field of W, see also Remark 3.3.5. This field is generated over K by $f_1, \ldots, f_r$ and since $f_1, \ldots, f_k$ are algebraically independent over K, we find $\operatorname{tr.deg}_K K(W) \geq k$. □

Remark 3.3.16 A closely related expression for the Kodaira dimension is the following, see [47, Sect.V.6]. For a nonsingular projective variety X over an algebraically closed field K we have

$$\kappa(X) = \operatorname{tr.deg}_K R - 1,$$

where R is the graded ring

$$R = \bigoplus_{n \geq 0} H^0(X, \omega_X^{\otimes n}).$$

We now have two different approaches to check if a variety is of general type, one using sheaf theory and one using concrete differentials. We will be using both in order to get a feeling of how to go from one approach to the other. We will start with a proposition about nonsingular varieties. Then the singularities will come in, first only on curves, then on a cone over a curve, the first surface. Then comes the surface Υ which will indeed turn out to be of general type and after that we will see some more general results about surfaces.

Proposition 3.3.17 *Let X be a nonsingular complete intersection in $\mathbb{P}^n_K$ of dimension ≥ 1, say of r hypersurfaces of degree $d_1, d_2, \ldots, d_r$. Let $\rho\colon X \hookrightarrow \mathbb{P}^n_K$ be the embedding of X in $\mathbb{P}^n_K$. Let ω_X be the canonical sheaf on X and set $m = -n-1+\sum_{i=1}^r d_i$, then ω_X is isomorphic to $\rho^*\mathcal{O}(m)$ and we have*

$$\kappa(X) = \begin{cases} -1 & \text{if } m < 0, \\ 0 & \text{if } m = 0, \\ \dim X = n-r & \text{if } m > 0. \end{cases}$$

For the proof of Proposition 3.3.17 we could refer to [60, exa.6.9.1] or in the case of surfaces, i.e., $r = n-2$, to [47, exc. V.6.1]. However, we will actually give two proofs for this proposition. The first is an easy consequence of lemmata and propositions involving sheaf theory that we have already seen. For the second proof, which uses concrete differentials, we will also first state several lemmata. This proof is more laborious, but also gives extra insight in the situation.

First proof of Proposition 3.3.17. Let ω_X and $\omega_{\mathbb{P}}$ denote the canonical sheaves on X and $\mathbb{P}^n$ respectively. From Proposition 3.1.27, we find that $\omega_X \cong \omega_{\mathbb{P}} \otimes \bigwedge^r \mathcal{N}_{X/\mathbb{P}^n}$, which by Proposition 3.1.36 is isomorphic to $\rho^*\mathcal{O}(m)$ with $m = -n-1+\sum_{i=1}^r d_i$. Lemma 3.3.10 finishes the proof. □

Lemma 3.3.18 *Let X be a subvariety of $\mathbb{A}^n_K$ of codimension k and suppose that X is a complete intersection, namely given by $f_1 = \ldots = f_k = 0$, with $f_1, \ldots, f_k \in K[x_1, \ldots, x_n]$. Then for each nonsingular point P on X there is a surjection $\bigwedge^{n-k} T_P^*\mathbb{A}^n_K \to \bigwedge^{n-k} T_P^* X$ and there is exactly one differential $\omega \in \bigwedge^{n-k} T_P^* X$ such that for any lift $\omega' \in \bigwedge^{n-k} T_P^*\mathbb{A}^n_K$ of ω we have $\omega' \wedge df_1 \wedge \cdots \wedge f_k = dx_1 \wedge \cdots \wedge dx_n \in \bigwedge^n T_P^*\mathbb{A}^n_K$.*

Proof. Let ψ be the submersion $\psi\colon \mathbb{A}^n_K \to \mathbb{A}^k_K\colon P \mapsto (f_1(P), \ldots, f_k(P))$ and set $Q = (0, \ldots, 0) \in \mathbb{A}^k_K$. Let $z_1, \ldots, z_k$ denote the coordinates of $\mathbb{A}^k$. Then we get $X = \psi^{-1}(Q)$ and for $P \in X$ we get an exact sequence

$$0 \to T_P X \to T_P\mathbb{A}^n \to T_Q\mathbb{A}^k \to 0.$$

Dualizing gives

$$0 \to T_Q^*\mathbb{A}^k \xrightarrow{\psi^*} T_P^*\mathbb{A}^n \to T_P^* X \to 0.$$

This implies that $\bigwedge^{n-k} T_P^*\mathbb{A}^n_K \to \bigwedge^{n-k} T_P^* X$ is indeed a surjection and by Lemma 3.1.37 we get an isomorphism

$$\varphi\colon \bigwedge^{n-k} T_P^* X \otimes \bigwedge^{k} T_Q^*\mathbb{A}^k \to \bigwedge^{n} T_P^*\mathbb{A}^n.$$

Since $\psi^* dz_i = df_i$ we find from Lemma 3.1.37 that for every $\omega \in \bigwedge^{n-k} T_P^* X$ and any lift $\omega' \in \bigwedge^{n-k} T_P^*\mathbb{A}^n_K$ of ω we have $\varphi(\omega \otimes (dz_1 \wedge \cdots \wedge dz_k)) = \omega' \wedge df_1 \wedge \cdots \wedge df_k$. Since $dz_1 \wedge \cdots \wedge dz_k \neq 0$ and $\bigwedge^n T_P^*\mathbb{A}^n$ is 1-dimensional, there is exactly one ω with $\varphi(\omega \otimes (dz_1 \wedge \cdots \wedge dz_k)) = dx_1 \wedge \cdots \wedge dx_n$. □

In the following Lemmata we will pose some concrete differentials that will turn out to be useful. We will start with some affine results, which we will then turn into projective results.

Definition 3.3.19 *Let polynomials $f_1, \ldots, f_k \in K[x_1, \ldots, x_n]$ and a sequence $J = (j_t)_{t=1}^k$ be given with $1 \leq j_1 < \ldots < j_k \leq n$. Then we define $M_J(f_1, \ldots, f_k)$ to be the determinant of the matrix*

$$A = \left(\frac{\partial f_i}{\partial x_{j_t}} \right)_{i,t=1}^k,$$

whence it is a $k \times k$-subdeterminant of the Jacobian matrix. If the polynomials f_i are understood from the context, then we write $M_J = M_J(f_1, \ldots, f_k)$.

Lemma 3.3.20 *In* $\bigwedge^k T_P^*\mathbb{A}^n$ *we have*

$$df_1 \wedge \cdots \wedge df_k = \sum_J M_J dx_{j_1} \wedge \cdots \wedge dx_{j_k},$$

where the sum is taken over all $\binom{n}{k}$ *sequences* $J = (j_t)_{t=1}^k$ *with* $1 \leq j_1 < \ldots < j_k \leq n$.

Proof. First note that $df_i = \sum_{j=1}^k \frac{\partial f_i}{\partial x_j} dx_j$, so by expanding it is clear that we can write $df_1 \wedge \ldots \wedge df_k = \sum_J C_J dx_{j_1} \wedge \cdots \wedge dx_{j_k}$ for some coefficients C_J. By induction we see that these coefficients can be given by

$$C_J = \sum_\sigma \operatorname{sign}(\sigma) \prod_{i=1}^k \frac{\partial f_{\sigma(i)}}{\partial x_{j_i}},$$

where the sum is taken over all the permutations $\sigma \in S_k$. This means that $C_J = M_J$. □

Lemma 3.3.21 *Let* $X \subset \mathbb{A}_K^n$ *be a variety of codimension* k. *Suppose that* X *is a complete intersection, say given by* $f_1 = \ldots = f_k = 0$ *with* $f_1, \ldots, f_k \in K[x_1, \ldots, x_n]$. *For a sequence* J *as in Definition 3.3.19 let* I *be the sequence* $I = (i_s)_{s=1}^{n-k}$ *with* $1 \leq i_1 < \ldots < i_s \leq n$ *such that* I *and* J *are disjoint. If we write* $\varepsilon(J)$ *for the sign of the permutation*

$$\begin{bmatrix} 1 & 2 & \ldots & n-k & n-k+1 & \ldots & n \\ i_1 & i_2 & \ldots & i_{n-k} & j_1 & \ldots & j_k \end{bmatrix}$$

then the differential

$$\omega_J = \frac{\varepsilon(J)}{M_J} dx_{i_1} \wedge \cdots \wedge dx_{i_{n-k}}$$

on X *is the same for each* J.

Proof. Using the fact that $dx_i \wedge dx_i = 0$ we find from Lemma 3.3.20 that for any lift $\omega'_J \in \bigwedge^{n-k} T_P^*\mathbb{A}_K^n$ of ω_J we have

$$\omega'_J \wedge df_1 \wedge \cdots df_k = dx_1 \wedge \cdots \wedge dx_n.$$

Lemma 3.3.18 says that this determines ω_J uniquely. □

Lemma 3.3.22 *Let* $X \subset \mathbb{A}_K^n$ *and* $f_1, \ldots, f_k$ *be as in Lemma 3.3.21. Let* $P = (x_1(P), \ldots, x_n(P))$ *be a nonsingular point on* X. *Then there exists a sequence* $J = (j_t)$ *as in Definition 3.3.19 and a corresponding sequence* $I = (i_s)$ *as in Lemma 3.3.21 such that* $M_J(P) \neq 0$ *and for* $s = 1, \ldots, n-k$ *the functions* $x_{i_s} - x_{i_s}(P)$ *form a set of local parameters.*

Proof. Since P is nonsingular, the rank of the $k \times n$ Jacobian matrix $(\partial f_l / \partial x_j)_{1 \leq l \leq k}^{1 \leq j \leq n}$ at P equals k, so there exists a $J = (j_t)_{t=1}^k$ with $1 \leq j_1 < \ldots < j_k \leq n$ such that the vectors $(\partial f_l / \partial x_{j_t})_l$ are linearly independent at P. Then $M_J(P) \neq 0$. By a linear transformation we can bring the corresponding $k \times k$ submatrix in triangular form. That is, there is an invertible matrix $A = (a_{r,t})_{r,t=1}^k$ such that for $y_r = \sum_{t=1}^k a_{r,t} x_{j_t}$ we have $\frac{\partial f_l}{\partial y_r}(P) = 0$ for $r < l$. This means that we can write

$$f_l = \sum_{r=l}^k c_{l,r}(y_r - y_r(P)) + \sum_{s=1}^{n-k} d_{l,s}(x_{i_s} - x_{i_s}(P)) + \text{h.o.t.} \tag{32}$$

with some constants $c_{l,r}$ and $d_{l,s}$ and where the higher order terms are monomials in the $y_r - y_r(P)$ and $x_{i_s} - x_{i_s}(P)$ of degree ≥ 2. Let $\mathcal{O}_{P,X}$ be the local ring at P and $\mathfrak{m}_P$ be its maximal ideal. Then $\mathfrak{m}_P/\mathfrak{m}_P^2$ is generated by the $x_j - x_j(P)$ and therefore also by $y_1 - y_1(P), \ldots, y_k - y_k(P)$ and

$x_{i_1} - x_{i_1}(P), \ldots, x_{i_{n-k}} - x_{i_{n-k}}(P)$. Since $\prod_{l=1}^{k} c_{l,l} = M_J(P) \neq 0$, we find $c_{l,l} \neq 0$ for all l, so using (32) we can write

$$y_l - y_l(P) = -c_{l,l}^{-1} \left(\sum_{r=l+1}^{k} c_{l,r}(y_r - y_r(P)) + \sum_{s=1}^{n-k} d_{l,s}(x_{i_s} - x_{i_s}(P)) \right) \pmod{\mathfrak{m}_P^2}.$$

This implies that in $\mathfrak{m}_P/\mathfrak{m}_P^2$ we can express the $y_l - y_l(P)$ inductively in the $x_{i_s} - x_{i_s}(P)$, so these form a set of local parameters. □

Corollary 3.3.23 *Let $X \subset \mathbb{A}_K^n$ and $\omega = \omega_J$ be as in Lemma 3.3.21. Then for any nonsingular point P there is a neighborhood U of P such that ω has no zeroes or poles in U. In particular, if X is nonsingular, then for the divisor (ω) on X we have $(\omega) = 0$.*

Proof. Let P be a nonsingular point, then by Lemma 3.3.22 there is a sequence $J = (j_t)$ as in Definition 3.3.19 and a corresponding sequence $I = (i_s)$ as in Lemma 3.3.21 such that $M_J(P) \neq 0$ and the $x_{i_s} - x_{i_s}(P)$ form a set of local parameters. Then there is an open neighborhood U of P such that for all $Q \in U$ we have $M_J(Q) \neq 0$ and the $x_{i_s} - x_{i_s}(Q)$ form a set of local parameters in Q. Since $dx_{i_s} = d(x_{i_s} - x_{i_s}(Q))$, this implies that the differential

$$\omega = \omega_J = \frac{\pm 1}{M_J} dx_{i_1} \wedge \cdots \wedge dx_{i_{n-k}}$$

has indeed no zeroes or poles in U. □

Knowing the behaviour of the differential $\omega = \omega_J$ as in Lemma 3.3.21 on the affine part of a projective variety X, we could check its behaviour at the hyperplane at infinity to find out its global behaviour. It will turn out that the contribution of the hyperplane H at infinity to the divisor (ω) is mH, where $m = -n - 1 + \sum \deg f_i$. As a matter of fact, for polynomials $g, h \in K[x_1, \ldots, x_n]$ the divisor of $\frac{g}{h}\omega$ on the regular locus of X is given by $(g) - (h) + (m - \deg g + \deg h)H$. In particular, taking $\frac{g}{h} = l^m$ for some linear polynomial, we get $m - \deg g + \deg h = 0$ and we find a result for projective varieties similar to Lemma 3.3.21. Projectively, this is stated as follows.

Lemma 3.3.24 *Let $X \subset \mathbb{P}_K^n$ be a variety of codimension k. Suppose that X is a complete intersection, say given by $F_1 = \ldots = F_k = 0$, with $F_1, \ldots, F_k \in K[X_0, X_1, \ldots, X_n]$ homogeneous polynomials. For a sequence $J = (j_t)_{t=1}^{k}$ with $0 \leq j_1 < \ldots < j_k \leq n$ let $M_J = M_J(F_1, \ldots, F_k)$ be the determinant of the matrix $(\partial F_l / \partial X_{j_t})_{l,t}$. For any i_0 not in J let $I = I(J, i_0)$ be the unique sequence $(i_s)_{s=1}^{n-k}$ with $0 \leq i_1 < \ldots < i_{n-k} \leq n$ such that i_0 is not in I and I and J are disjoint. If we choose a linear form $L \in K[X_0, \ldots, X_n]$, set $m = -n - 1 + \sum_{i=1}^{k} \deg F_i$ and we write $\varepsilon(J, i_0)$ for the sign of the permutation*

$$\begin{bmatrix} 0 & 1 & \ldots & n-k & n-k+1 & \ldots & n \\ i_0 & i_1 & \ldots & i_{n-k} & j_1 & \ldots & j_k \end{bmatrix},$$

then the differential

$$\omega_{J,i_0} = \frac{\varepsilon(J, i_0) X_{i_0}^{n-k+1} L^m}{M_J} d\left(\frac{X_{i_1}}{X_{i_0}}\right) \wedge \cdots \wedge d\left(\frac{X_{i_{n-k}}}{X_{i_0}}\right) \tag{33}$$

is the same for each J and i_0 not in J. For each point P there is some neighborhood U on which the divisor (ω_{J,i_0}) is given by $m(H \cap U)$, where H is the hyperplane determined by $L = 0$.

Proof. Note that the degree of M_J is equal to $\sum(\deg F_i - 1) = m + n + 1 - k$, so (33) gives a well defined differential. From

$$d\left(\frac{X_{i_l}}{X_{i_0}}\right) = -\frac{X_{i_l}^2}{X_{i_0}^2} d\left(\frac{X_{i_0}}{X_{i_l}}\right)$$

and

$$d\left(\frac{X_{i_s}}{X_{i_0}}\right) = \frac{X_{i_l}}{X_{i_0}} d\left(\frac{X_{i_s}}{X_{i_l}}\right) - \frac{X_{i_l}X_{i_s}}{X_{i_0}^2} d\left(\frac{X_{i_0}}{X_{i_l}}\right)$$

we conclude that

$$d\left(\frac{X_{i_1}}{X_{i_0}}\right)\wedge\cdots\wedge d\left(\frac{X_{i_{n-k}}}{X_{i_0}}\right) = \varepsilon\frac{X_{i_l}^{n-k+1}}{X_{i_0}^{n-k+1}}\cdot d\left(\frac{X_{i_0}}{X_{i_l}}\right)\wedge\cdots\wedge d\left(\frac{X_{i_{l-1}}}{X_{i_l}}\right)\wedge d\left(\frac{X_{i_{l+1}}}{X_{i_l}}\right)\wedge\cdots\wedge d\left(\frac{X_{i_{n-k}}}{X_{i_l}}\right),$$

where ε is the sign of the permutation

$$\begin{bmatrix} i_0 & i_1 & \dots & i_l & i_{l+1} & \dots & i_{n-k} \\ i_l & i_0 & \dots & i_{l-1} & i_{l+1} & \dots & i_{n-k} \end{bmatrix}.$$

In combination with Lemma 3.3.21 it follows that $\omega(J, i_0)$ is indeed the same for each J and i_0 not in J. Hence we can write $\omega = \omega(J, i_0)$. Each point P on X is contained in an open affine given by $X_{i_0} \neq 0$ for some i_0. Just as in the proof of Corollary 3.3.23 we now find that if P is nonsingular, then there is an open neighborhood U of P such that $(\omega)|_U = m(H \cap U)$. □

Corollary 3.3.25 *Let $X \subset \mathbb{P}^n_K$ be a variety of codimension k. Suppose that X is a complete intersection, say given by $F_1 = \ldots = F_k = 0$, with $F_1, \ldots, F_k \in K[X_0, X_1, \ldots, X_n]$ homogeneous polynomials of degree $\deg F_l = d_l$. Let ρ denote the embedding of the regular locus X^{reg} of X in $\mathbb{P}^n$, so $\rho\colon X^{\mathrm{reg}} \hookrightarrow \mathbb{P}^n$ and set $m = -n - 1 + \sum_{l=1}^k d_l$. Then the canonical sheaf $\omega_{X^{\mathrm{reg}}}$ of X^{reg} is isomorphic to $\rho^*\mathcal{O}(m)$.*

Proof. Let P be any nonsingular point. Then from Lemma 3.3.24 we find that there is an open neighborhood U of P such that if $H \subset \mathbb{P}^n$ is a hyperplane, then $m\rho^*H \cap U$ is a canonical divisor on U. As these open sets cover X^{reg} it follows that $m\rho^*H$ is a canonical divisor on X^{reg}. Therefore the canonical sheaf of X^{reg} is isomorphic to the sheaf $\mathcal{L}(m\rho^*H)$ on X^{reg}, which is isomorphic to $\rho^*\mathcal{O}(1)^m \cong \rho^*\mathcal{O}(m)$. □

Second Proof of Proposition 3.3.17. If X is nonsingular, then $X^{\mathrm{reg}} = X$, so from Corollary 3.3.25 we find that the canonical sheaf ω_X is isomorphic to $\rho^*\mathcal{O}(m)$. Lemma 3.3.10 finishes the proof. □

We now know how to compute the Kodaira dimension of nonsingular projective complete intersections. The next step is to consider singular varieties. We will start with curves, for which the Kodaira dimension will turn out to depend only on the genus. In general some inequalities about the m-genus imply inequalities involving the Kodaira dimension.

Lemma 3.3.26 *Let X be an algebraic variety over K. Then for the Kodaira dimension we have $\kappa(X) > 0$ if and only if for some positive integer m the inequality $p_m(X) \geq 2$ holds for the m-genus. We have $p_m(X) = 0$ for all m if and only if $\kappa(X) = -1$.*

Proof. Let $\widetilde{X}$ be a nonsingular projective surface over K that is birationally equivalent with X and suppose m is a positive integer such that $\dim_K H^0(\widetilde{X}, \omega_K^{\otimes m}) \geq 2$. Let $\omega_1, \omega_2 \in H^0(\widetilde{X}, \omega_K^{\otimes m})$ be two linearly independent differentials. Then for some nonconstant rational function we can write $\omega_1 = f\omega_2$. Since X is by definition geometrically irreducible, the algebraic closure of K in the field $K(X)$ of rational functions on X is equal to K itself. Hence the nonconstant rational function f is transcendental over K, so the m-canonical map of $\widetilde{X}$ is nonconstant and $\kappa(X) = \kappa(\widetilde{X}) > 0$. Conversely, if $\kappa(\widetilde{X}) > 0$, then there exists a positive integer m such that the transcendence degree of $H^0(X, \omega_X^{\otimes m})$ is at least 2, whence $p_m(X) \geq 2$. We have $p_m(X) = 0$ for all m if and only if $\mathbb{N}(\omega_X, X) = \emptyset$, so if and only if $\kappa(X) = -1$. □

Corollary 3.3.27 *Let C be a curve. Then the Kodaira dimension depends only on the (geometric) genus. If we have $p_g(C) = 0$, $p_g(C) = 1$ or $p_g(C) \geq 2$, then we have $\kappa(C) = -1$, $\kappa(C) = 0$, $\kappa(C) = 1$ respectively.*

Proof. Since $\kappa(C) \leq \dim C = 1$, the first statement of Lemma 3.3.26 is equivalent with the fact that $\kappa(C) = 1$ if and only if $p_m(C) \geq 2$ for some integer m. Hence if $p_g(C) = p_1(C) \geq 2$, then $\kappa(C) = 1$. If $p_g(C) = 0$, then C is (geometrically) isomorphic to $\mathbb{P}_1$ and a canonical divisor is given by $K_C = -2(Q)$ for any point Q. Therefore $mK_C \leq K_C$ for positive integers m and we get $H^0(C, \omega_C^{\otimes m}) = 0$. This implies that $p_m(C) = 0$ for all m, so from Lemma 3.3.26 we find $\kappa(C) = -1$. Finally, if $p_g(C) = 1$, then C is an elliptic curve (over some field extension). On an elliptic curve there exists a regular nonvanishing differential ω, so a canonical divisor K_C can be given by $K_C = (\omega) = 0$. Hence $mK_C = K_C$ for all m and therefore $p_m(C) = p_1(C) = p_g(C) = 1$ Again from Lemma 3.3.26 it follows that $\kappa(C) = 0$. □

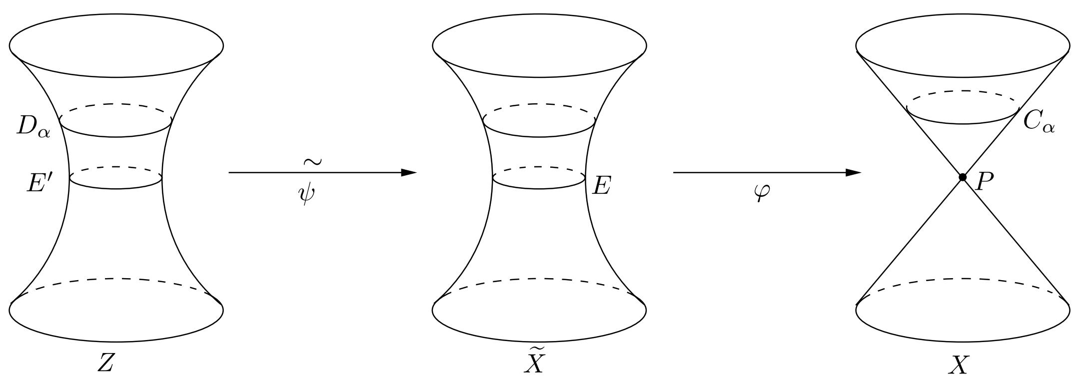
From the Theorem of Riemann-Hurwitz we know that if $C \to D$ is a finite separable morphism of curves, then the inequality $p_g(C) \geq p_g(D)$ holds. It actually holds in more generality and by Corollary 3.3.27 it implies that $\kappa(C) \geq \kappa(D)$, which also holds in a far more general situation.

Lemma 3.3.28 *Let* $f: X \to Y$ *be a generically surjective rational map of algebraic varieties such that* $\dim X = \dim Y$. *Then we have* $\kappa(X) \geq \kappa(Y)$.

Proof. See [60, Thm.II.6.10]. □

Remark 3.3.29 A variety is called *unirational* if it is a rational image of projective space. From Proposition 3.3.17 we know that $\kappa(\mathbb{P}^n) = -1$ for all n. Together with Lemma 3.3.28 this implies that for any unirational variety X we have $\kappa(X) = -1$.

For a nonsingular ruled surface X we also have $\kappa(X) = -1$, see [47, Thm.V.6.1.] or [35, Thm. VI.1.1.]. An example of a ruled surface is a cone over a curve.



Example 3.3.30 Let C be a nonsingular curve in $\mathbb{P}^2_K$ given by $F = 0$ for some homogeneous polynomial $F \in K[X, Y, Z]$ of degree e. Using the inclusion $K[X, Y, Z] \hookrightarrow K[X, Y, Z, W]$ we can define the cone $X \subset \mathbb{P}^3$ over C given by $F = 0$, see also [47, exc.I.2.10]. It is normal because of Proposition 3.2.17 and has a vertex $P = [x_P : y_P : z_P : w_P] = [0 : 0 : 0 : 1]$, which is the only singular point of X. Let $\varphi: \widetilde{X} \to X$ be the blow-up of X in the point P and set $E = \varphi^{-1}(P)$. The surface $\widetilde{X}$ can be given in $\mathbb{P}^3_K \times \mathbb{P}^2_K$ by

$$F(R, S, T) = 0, \quad XT = ZR, \quad YR = XS, \quad YT = ZS,$$

where X, Y, Z, W are the homogeneous coordinates of $\mathbb{P}^3$ and R, S, T are the homogeneous coordinates of $\mathbb{P}^2$. Then φ is given by projection on $\mathbb{P}^3$.

For $\alpha \in K^*$ let C_α denote the intersection of X with the hyperplane $H \subset \mathbb{P}^3$ given by $Z - \alpha W$. Then C_α is a nonsingular curve not passing through P which is isomorphic to C, whence $\varphi^{-1}(C_\alpha)$ is isomorphic to C_α. We will show that as a divisor on $\widetilde{X}$ it is linearly equivalent with E using the Segre-embedding, see [47, exc.I.2.14].

The Segre-embedding gives an embedding of $\widetilde{X}$ in $\mathbb{P}^{11}$. Using the fact that $XS = YR$, $XT = ZR$ and $YT = ZS$ we can project $\mathbb{P}^{11}$ onto $\mathbb{P}^8$ to get an embedding of $\widetilde{X}$ into $\mathbb{P}^8$. After some calculations we find that its image Z in $\mathbb{P}^8$ can be given by

$$\begin{array}{llll}
b_{01}b_{31} = b_{03}b_{11}, & b_{01}b_{33} = b_{03}b_{31}, & b_{22}b_{33} = b_{23}^2, & F(b_{01}, b_{02}, b_{03}) = 0, \\
b_{01}b_{22} = b_{02}b_{12}, & b_{02}b_{23} = b_{03}b_{22}, & b_{11}b_{33} = b_{31}^2, & F(b_{11}, b_{12}, b_{31}) = 0, \\
b_{01}b_{23} = b_{03}b_{12}, & b_{02}b_{33} = b_{03}b_{23}, & b_{12}b_{23} = b_{22}b_{31}, & F(b_{12}, b_{22}, b_{23}) = 0, \\
b_{01}b_{23} = b_{02}b_{31}, & b_{11}b_{22} = b_{12}^2, & b_{12}b_{33} = b_{23}b_{31}, & F(b_{31}, b_{23}, b_{33}) = 0, \\
b_{01}b_{12} = b_{02}b_{11}, & b_{11}b_{23} = b_{31}b_{12}, & &
\end{array}$$

where $b_{01}, b_{02}, b_{03}, b_{11}, b_{12}, b_{22}, b_{23}, b_{31}, b_{33}$ are homogeneous coordinates of $\mathbb{P}^8$. The isomorphism $\widetilde{X} \to Z$ is given by

$$[x : y : z : w] \times [r : s : t] \mapsto [wr : ws : wt : xr : xs : ys : yt : zr : zt]$$

Its inverse will be denoted ψ and can locally be given by

$$[b_{01} : b_{02} : b_{03} : b_{11} : b_{12} : b_{22} : b_{23} : b_{31} : b_{33}] \mapsto \left\{ \begin{array}{l} [b_{11} : b_{12} : b_{31} : b_{01}], \\ [b_{12} : b_{22} : b_{23} : b_{02}], \\ [b_{31} : b_{23} : b_{33} : b_{03}] \end{array} \right. \times \left\{ \begin{array}{l} [b_{01} : b_{02} : b_{03}], \\ [b_{11} : b_{12} : b_{31}], \\ [b_{12} : b_{22} : b_{23}], \\ [b_{31} : b_{23} : b_{33}] \end{array} \right.$$

It is easy to check that Z, and hence $\widetilde{X}$, is nonsingular. The curve E is isomorphic to $E' = \psi^{-1}(E)$, which is given by $b_{11} = b_{12} = b_{22} = b_{23} = b_{31} = b_{33} = 0$ on Z. On $\mathbb{P}^8$ the radical ideal $I(E')$ of E' is generated by

$$b_{11}, b_{12}, b_{22}, b_{23}, b_{31}, b_{33}, F(b_{01}, b_{02}, b_{03}).$$

Hence the projection on the b_{01}, b_{02} and b_{03} coordinates induces an isomorphism between E' and the curve C. These facts also follow from [47, exc.I.5.7].

For $\alpha \in K^*$ define $D_\alpha = \psi^{-1}\varphi^{-1}(C_\alpha)$, then D_α is isomorphic to C_α. On Z the curve D_α is given by $b_{33} - \alpha b_{03} = b_{23} - \alpha b_{02} = b_{31} - \alpha b_{01} = 0$. On $\mathbb{P}^8$ the radical ideal $I(D_\alpha)$ is equal to the ideal I_α generated by

$$\begin{array}{llll}
b_{33} - \alpha b_{03}, & b_{11}b_{22} - b_{12}^2, & b_{12}b_{31} - b_{11}b_{23}, & F(b_{01}, b_{02}, b_{03}), \\
b_{23} - \alpha b_{02}, & b_{22}b_{33} - b_{23}^2, & b_{23}b_{12} - b_{22}b_{31}, & F(b_{11}, b_{12}, b_{31}), \\
b_{31} - \alpha b_{01}, & b_{33}b_{11} - b_{31}^2, & b_{31}b_{23} - b_{33}b_{12}, & F(b_{12}, b_{22}, b_{23}).
\end{array}$$

The ideal $I_{\alpha=0}$ is equal to $I(X) + (b_{33})$, which is the intersection of two radical ideals, namely $I(E')$ and J, where J is the ideal generated by

$$\begin{array}{llll}
b_{03}, & b_{23}, & b_{11}b_{22} - b_{12}^2, & F(b_{01}, b_{02}, b_{03}), \\
b_{31}, & b_{33}, & b_{22}b_{33} - b_{23}^2, & F(b_{11}, b_{12}, b_{31}), \\
 & & b_{33}b_{11} - b_{31}^2, & F(b_{12}, b_{22}, b_{23})
\end{array}$$

and its zeroset has dimension zero. For $\alpha \neq 0$ we also happen to have $I(X) + (b_{33} - \alpha b_{03}) = I_\alpha + J$. It follows that for $\alpha \neq 0$ and $f = (b_{33} - \alpha b_{03})b_{33}^{-1} = 1 - \alpha b_{03}/b_{33}$ the principal divisor (f) on Z equals $D_\alpha - E'$, so E' is linear equivalent with D_α. Hence on $\widetilde{X}$ we now know that E is indeed linearly equivalent with $\varphi^{-1}(C_\alpha)$.

Let $L \in K[X, Y, Z, W]$ be homogeneous linear polynomial with $L(P) \neq 0$ and let $H \subset \mathbb{P}^3$ be the hyperplane given by $L = 0$ and write $U = X - \{P\}$. By Lemma 3.3.24 the divisor of the differential

$$\omega = \frac{W^3 L^{e-4}}{\frac{\partial F}{\partial X}} d\left(\frac{Y}{W}\right) \wedge d\left(\frac{Z}{W}\right)$$

equals $(\omega)|_U = (e-4)(H \cap U)$, but to compute the Kodaira dimension we need to look at $\widetilde{X}$ or Z. Since U is isomorphic to $\varphi^{-1}(U) = \widetilde{X} - E$, we only need to find out the contribution of E to

$(\varphi^*\omega)$, so we can restrict to the affine part $W \neq 0$ of $\mathbb{P}^3$. We first look at the affine part $T \neq 0$ of $\mathbb{P}^2$. Hence we are looking at $\mathbb{A}^3 \times \mathbb{A}^2 \cong \mathbb{A}^5$ with coordinates x, y, z, r, s on which $\widetilde{X}$ is given by $y = zs$, $x = zr$ and $F(r, s, 1) = 0$. On this affine part we have $\partial F/\partial x = z^{e-1}\partial F(r, s, 1)/\partial r$, so $\varphi^*\omega$ is given by

$$\varphi^*\omega = \frac{l^{e-4}}{\frac{\partial F}{\partial x}} d(zs) \wedge dz = \frac{l^{e-4}z}{\frac{\partial F}{\partial x}} ds \wedge dz = \frac{l^{e-4}}{z^{e-2}\frac{\partial F(r,s,1)}{\partial r}} ds \wedge dz = \frac{l^{e-4}}{z^{e-2}\frac{\partial F(r,s,1)}{\partial s}} dr \wedge dz.$$

Here we put $l = L(x, y, z, 1)$. Since $E \cong C$ is nonsingular, we have

$$\frac{\partial F(r, s, 1)}{\partial r}(P) = 0 \quad \text{or} \quad \frac{\partial F(r, s, 1)}{\partial s}(P) = 0$$

for every $P \in E$. Since E is given by $z = 0$ on this affine part, this implies that in a neighborhood V of this affine part of E we get $(\varphi^*\omega)|_V = (e - 4)(\varphi^*H \cap V) + (2 - e)(E \cap V)$. Looking at the affine parts $R \neq 0$ and $S \neq 0$ we find similar results, which after putting together imply that $(\varphi^*\omega) = (e - 4)\varphi^*H + (2 - e)E$. Since E is linear equivalent with φ^*H and $\mathcal{L}(\varphi^*H)$ is isomorphic to $\varphi^*\mathcal{O}(1)$, this implies that the canonical sheaf on $\widetilde{X}$ is isomorphic to $\varphi^*\mathcal{O}(-2)$. Since $\varphi^*\mathcal{O}(2)$ is pseudo-ample, it follows from Lemma 3.3.9, that $\kappa(X) = \kappa(\varphi^*\mathcal{O}(-2), X) = -1$.

Similar as in this example we can compute the Kodaira dimension of Υ using concrete differentials.

Proposition 3.3.31 *The surface Υ is of general type. The canonical sheaf on $\widetilde{\Upsilon}$ is isomorphic to $\pi^*\mathcal{O}(1)$.*

Proof. Since Υ is of general type if and only if $\widetilde{\Upsilon}$ is, it suffices by Lemma 3.3.8 to show that the canonical sheaf on $\widetilde{\Upsilon}$ is isomorphic to $\mathcal{L}(\pi^*H)$ for any hyperplane $H \subset \mathbb{P}^6_{\overline{\mathbb{Q}}}$ as we have $\mathcal{L}(\pi^*H) \cong \pi^*\mathcal{L}(H) \cong \pi^*\mathcal{O}(1)$. Hence it suffices to show that for any linear form $L \in \overline{\mathbb{Q}}[A, B, C, X, Y, Z, U]$ such that $L(R_i) \neq 0$ for any singular point R_i, the differential

$$\omega = \frac{A^3L}{CYZU} d\left(\frac{X}{A}\right) \wedge d\left(\frac{B}{A}\right) \tag{34}$$

on Υ induces a differential $\pi^*\omega$ on $\widetilde{\Upsilon}$ which is regular everywhere.

We could check easily that ω is regular on the open sets $\tau^k U_i$, with $i, k = 1, 2, 3$, described in section 3.2. However, since ω is the differential described in (33), Lemma 3.3.24 already tells us that on the regular locus $U = \Upsilon^{\text{reg}}$ of Υ we have $(\omega)|_U = H \cap U$, where $H \subset \mathbb{P}^6_{\overline{\mathbb{Q}}}$ is the hyperplane given by $L = 0$. Hence we only need to check that $\pi^*\omega$ on $\widetilde{\Upsilon}$ is regular in a neighborhood of $\pi^{-1}(R_i)$ for the singular points R_i. Since $G \subset \text{Aut}(\Upsilon)$ acts transitively on the singular points and $g(\omega) = \pm\frac{g(L)}{L}\omega$ for all $g \in G$, it suffices to do this for $R_1 = [1 : 0 : 0 : 0 : 1 : 1 : 1]$ only.

We will look at the affine part $A \neq 0$ on which ω can be written as

$$\omega = \frac{l}{cyxu} dx \wedge db.$$

The point R_1 is contained in the open set U given by $yzu \neq 0$. The point R_1 on U locally looks like the vertex $(0, 0, 0)$ on the surface in $\mathbb{A}^3_{\overline{\mathbb{Q}}}$ given by $f = 0$ with $f = b^2 + c^2 - x^2$. Just as in Example 3.3.30 we now find that the contribution of E_1 on $\widetilde{\Upsilon}$ to the divisor $(\pi^*\omega)$ is $(2 - \deg f)E_1 = 0$. Hence on $\widetilde{\Upsilon}$ we find $(\pi^*\omega) = \pi^*H$. It follows that the canonical sheaf is indeed isomorphic to $\pi^*\mathcal{L}(H) \cong \pi^*\mathcal{O}(1)$ and by Lemma 3.3.8 we find for the Kodaira dimension that $\kappa(\Upsilon) = \kappa(\widetilde{\Upsilon}) = 2$. □

Remark 3.3.32 We have seen that Proposition 3.3.31 follows from regularity of $\pi^*\omega$ on $\widetilde{\Upsilon}$ for ω as in (34). This also follows from Lemma 3.3.15. Indeed, we can choose 2 other linear forms $L', L'' \in \overline{\mathbb{Q}}[A,B,C,X,Y,Z,U]$ such that L'/L and L''/L are algebraically independent. Then the differentials $\omega, \omega' = \frac{L'}{L}\omega$ and $\omega'' = \frac{L''}{L}\omega$ are $3 = \dim \widetilde{\Upsilon} + 1$ algebraically independent regular differential forms of degree 2 on $\widetilde{\Upsilon}$, so the Kodaira dimension equals $\dim \widetilde{\Upsilon}$.

Corollary 3.3.33 *Let $K_{\widetilde{\Upsilon}}$ be a canonical divisor on $\widetilde{\Upsilon}$. Then we have $K_{\widetilde{\Upsilon}}^2 = 16$.*

Proof. Let $H \subset \mathbb{P}^6_{\overline{\mathbb{Q}}}$ be a hyperplane that does not contain any of the 48 singular points of Υ. Then $D' = \pi^* H$ is a canonical divisor on $\widetilde{\Upsilon}$, whence D' is linearly equivalent with $K_{\widetilde{\Upsilon}}$ and $D'^2 = K_{\widetilde{\Upsilon}}^2$. Write $D = H \cap \Upsilon$. Since H does not contain any of the singular points and π induces an isomorphism $\widetilde{\Upsilon} - \bigcup_{j=1}^{48} E_j \to \Upsilon^{\text{reg}} = \Upsilon - \{R_j : 1 \le j \le 48\}$ we find that $D'^2 = D^2$, where the first is a self intersection number on $\widetilde{\Upsilon}$ and the second on Υ.

Let $H' \subset \mathbb{P}^6$ be another hyperplane and let H and H' be such that $H \cap H'$ intersects Υ transversally. Since D is linearly equivalent with $H' \cap \Upsilon$ we find $D^2 = \deg(H \cap \Upsilon) \cap (H' \cap \Upsilon) = \deg(H \cap H') \cap \Upsilon$. Since $H \cap H'$ is a line intersecting Υ transversally, it intersects Υ by Bézout's Theorem in $\deg \Upsilon = 16$ points, so $D^2 = 16$ and $K_{\widetilde{\Upsilon}}^2 = D'^2 = D^2 = 16$. □

Corollary 3.3.34 *The topological Euler characteristic $\chi_{\text{top}}(\widetilde{\Upsilon}(\mathbb{C}))$ equals 80. For the Hodge number $h^{1,1}$ of $\widetilde{\Upsilon} \otimes \mathbb{C}$ we have $h^{1,1}(\widetilde{\Upsilon} \otimes \mathbb{C}) = 64 + 2q$, where $q = h^{1,0}$ is as in Proposition 3.2.19.*

Proof. Remember that $\chi(\widetilde{\Upsilon}) = 8$, so from the Noether Formula 3.1.52 and Corollary 3.3.33 we find

$$\chi_{\text{top}}(\widetilde{\Upsilon}(\mathbb{C})) = 12\chi(\widetilde{\Upsilon} \otimes \mathbb{C}) - K_{\widetilde{\Upsilon}}^2 = 12 \cdot 8 - 16 = 80.$$

From the Hodge numbers already computed in Proposition 3.2.19 we find that $80 = \chi_{\text{top}}(\widetilde{\Upsilon}(\mathbb{C})) = 1 - 2q + (14 + 2q + h^{1,1}) - 2q + 1 = 16 + h^{1,1} - 2q$, from which $h^{1,1}$ follows. □

Bluff 2 *We actually have $h^{1,1}(\widetilde{\Upsilon} \otimes \mathbb{C}) = 64$.*

Proof. From Bluff 1 we know that $h^{1,0} = q = 0$. Hence it follows from Corollary 3.3.34 that $h^{1,1} = 64$. □

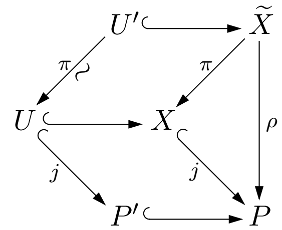
Because Υ is a complete intersection which is nonsingular except for some isolated singular points which are ordinary double points, the fact that Υ is of general type will also follow from proposition 3.3.35 together with Lemma 3.3.8 for then we have $e_j = 2$ for all j.

First let us set some notation. Let $X \subset \mathbb{P}^n$ be a surface over K which is a complete intersection of $n-2$ hypersurfaces of degree $d_1, \ldots, d_{n-2}$. Suppose that X is nonsingular except for some isolated singular points $Q_1, \ldots, Q_t$. Suppose also that for all $j = 1, \ldots, t$ there is a nonsingular curve $C_j \subset \mathbb{P}^n$ of degree e_j such that Q_j is analytically isomorphic to the vertex of the cone over C_j. Let j denote the inclusion $j \colon X \hookrightarrow \mathbb{P}^n$ and let $U = X^{\text{reg}}$ be the maximal regular open subset of X, i.e., $X^{\text{reg}} = X \setminus X^{\text{sing}}$, where X^{sing} is the singular locus of X. Let $\pi \colon \widetilde{X} \to X$ be the blow-up of X in the points Q_j. Let $E_j = \pi^{-1}(Q_j)$ be the fibre of π above Q_j. Let H be a hyperplane of $\mathbb{P}^n$ that does not contain any of the Q_j. This is possible since we can work over an algebraically closed, whence infinite, field. Define $U' := \pi^{-1}(U)$ and $\rho = j \circ \pi$. Finally write $P := \mathbb{P}^n$, $P' = \mathbb{P}^n \setminus X^{\text{sing}}$ and $m = -1 - n + \sum_{i=1}^{n-2} d_i$.

Proposition 3.3.35 *Under the assumptions above the surface $\widetilde{X}$ is regular and a canonical divisor on $\widetilde{X}$ is given by*

$$m\rho^* H + \sum_{j=1}^{t} (2 - e_j) E_j.$$

We have the following diagram and will first compute the canonical sheaf on U' via those of U, P' and P. Then we will compute a canonical divisor on $\widetilde{X}$ from the canonical sheaf on U'.



Lemma 3.3.36 *The surface $\widetilde{X}$ is regular, E_j is isomorphic to C_j, has genus $g_j = \frac{1}{2}(e_j - 1)(e_j - 2)$ and self intersection number $E_j^2 = -e_j$.*

Proof. Apart from the genus g_j this is just a restatement of Lemma 3.2.8. Hence the genus g_j of E_j is equal to that of C_j. Since C_j is a nonsingular degree e_j curve embedded in $\mathbb{P}^2$, the genus of C_j indeed equals $\frac{1}{2}(e_j - 1)(e_j - 2)$. □

Lemma 3.3.37 *Let ω_U be the canonical sheaf on the nonsingular variety $U = X^{\text{reg}}$. Then we have an isomorphism $\omega_U \cong j^*\mathcal{O}(m)$ of sheaves on U.*

Proof. Let $\mathcal{I}$ be the idealsheaf of X in P and $\mathcal{I}'$ the ideal sheaf of U in P'. Let $\omega_{P'}$ and ω_P denote the canonical sheaves on P' and P respectively. Then $\omega_P|_{P'} \cong \omega_{P'}$ and $\mathcal{I}|_{P'} \cong \mathcal{I}'$. It follows that

$$(\omega_P \otimes \bigwedge^{n-2}(\mathcal{I}/\mathcal{I}^2)^\vee)|_U \cong \omega_{P'} \otimes \bigwedge^{n-2}(\mathcal{I}'/\mathcal{I}'^2)^\vee.$$

By taking $Y = U$ and $X = P'$ for the X and Y in Proposition 3.1.27 we find that the right hand side is isomorphic to ω_U. The left hand side is isomorphic to $j^*\mathcal{O}(m)|_U$ by Lemma 3.1.36. □

Lemma 3.3.38 *Let $\omega_{U'}$ be the canonical sheaf on the nonsingular variety U'. Then we have an isomorphism $\omega_{U'} \cong \rho^*\mathcal{O}(m)$ of sheaves on U'.*

Proof. Since π induces an isomorphism from U' to U, we find from Lemma 3.3.37 and the fact that $\rho^* = \pi^* \circ j^*$ that

$$\omega_{U'} \cong \pi^*\omega_U \cong \pi^* j^*\mathcal{O}(m) = \rho^*\mathcal{O}(m).$$

□

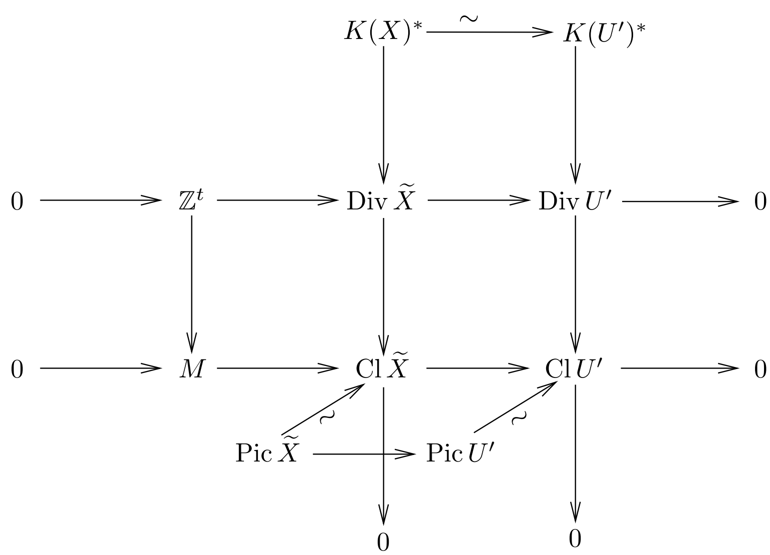
Before we can compute the canonical divisor on $\widetilde{X}$, we need the following propositions. Remember that for a noetherian integral separated scheme Y which is regular in codimension one the group $\operatorname{Div} Y$ is the group of divisors on Y and that $\operatorname{Cl} Y$ is the group of divisor classes. Remember also that if Y is also factorial, then the group $\operatorname{Cl} Y$ is isomorphic to the group $\operatorname{CaCl} Y$ of Cartier divisors and hence with the group $\operatorname{Pic} Y$ of invertible sheaves.

Proposition 3.3.39 (Adjunction Formula) *If C is a nonsingular curve of genus g_C on a nonsingular surface S, and if K_S is a canonical divisor on S, then*

$$2g_C - 2 = C \cdot (C + K_S).$$

Proof. See Hartshorne [Proposition V.1.5]. □

Proof of Proposition 3.3.35. Since $\widetilde{X}$ is a nonsingular variety we can apply Proposition 3.1.5 and it tells us that there is a surjective homomorphism $\operatorname{Cl} \widetilde{X} \to \operatorname{Cl} U'$. Let M denote the kernel. The 48 exceptional curves E_j of codimension 1 are irreducible, whence prime divisors, so we find that there is an exact sequence which is the top row of the following commutative diagram in which the two rows and the two columns are exact. Note that the top two vertical arrows are not injections.



Let $K_{\widetilde{X}} \in \operatorname{Div} \widetilde{X}$ be a (canonical) divisor the class of which corresponds with the sheaf $\omega_{\widetilde{X}}$ and let $H \subset \mathbb{P}^n$ be a hyperplane that does not contain any singular point of X. Then the class of $m\rho^*H \in \operatorname{Div} \widetilde{X}$ in $\operatorname{Cl} \widetilde{X}$ corresponds with $\rho^*\mathcal{O}(m)$. The two sheaves $\omega_{\widetilde{X}}, \rho^*\mathcal{O}(m) \in \operatorname{Pic} \widetilde{X}$ have the same images in $\operatorname{Pic} U'$, whence also in $\operatorname{Cl} U'$. Therefore the image $[D|_{U'}]$ in $\operatorname{Cl} U'$ of $D := K_{\widetilde{X}} - m\rho^*H \in \operatorname{Div} \widetilde{X}$ is 0. Hence the image of D in $\operatorname{Div} U'$ is principal, say $D \cap U = (f)$ for some rational function $f \in K(U')$. Since U' and $\widetilde{X}$ are birationally equivalent, they have the same function field, so we can view f as an element of $K(\widetilde{X})$. The divisor $D - (f) \in \operatorname{Div} \widetilde{X}$ maps to 0 in $\operatorname{Div} U'$, so from the exactness of the top row of the diagram we find that we can write it as $D - (f) = \sum_j r_j E_j$ for certain integers r_j. This means that for $K'_{\widetilde{X}} := K_{\widetilde{X}} - (f)$ we have

$$K'_{\widetilde{X}} = m\rho^*H + \sum_{j=1}^{t} r_j E_j.$$

Since $K'_{\widetilde{X}}$ is also a canonical divisor we can use Proposition 3.3.39 to compute the r_j. Note that the E_j are isomorphic to C_j by Lemma 3.3.36 and that they have genus $g_j = \frac{1}{2}(e_j - 1)(e_j - 2)$. Substituting $C = E_i$ and $K_S = K'_{\widetilde{X}}$ in the adjunction formula, we find

$$e_j(e_j - 3) = 2g_i - 2 = C \cdot (C + K'_{\widetilde{X}}) = E_i \cdot (E_i + \rho^*(mH) + \sum_{j=1}^{t} r_j E_j). \qquad (35)$$

Since $E_i \cap \rho^*(H) = \emptyset$ and $E_i \cap E_j = \emptyset$ for $i \neq j$, we get $E_i \cdot m\rho^*(H) = E_i \cdot E_j = 0$ for $i \neq j$. Therefore (35) simplifies to $e_j(e_j - 3) = E_i^2 + r_i E_i^2 = (1 + r_i)E_i^2$. Lemma 3.2.8 says that $E_i^2 = -e_i$, so $r_i = 2 - e_i$ and $K_{\widetilde{X}}$ is linearly equivalent with the canonical divisor $K'_{\widetilde{X}} = \rho^*(mH) + \sum_{i=1}^{t}(2 - e_j)E_j$. □

Corollary 3.3.40 *Let $X \subset \mathbb{P}^n_K$ be a projective surface which is a complete intersection, say X is the intersection of $n - 2$ hypersurfaces of degree $d_1, \ldots, d_{n-2}$. Suppose that X is regular except maybe for some isolated singular points all of which are ordinary double points. Set $m =$*

$-n-1+\sum d_i$, *then the Kodaira dimension of* X *equals*

$$\kappa(X) = \begin{cases} -1 & \textit{if } m < 0, \\ 0 & \textit{if } m = 0, \\ 2 & \textit{if } m > 0. \end{cases}$$

Proof. Let $\pi\colon \widetilde{X} \to X$ be the blow-up of X in the singular points. Let $H \subset \mathbb{P}^n$ be a hyperplane not containing any of the singular points. From Proposition 3.3.35 we find that $\widetilde{X}$ is nonsingular and that $m\pi^*H$ is a canonical divisor on $\widetilde{X}$, as all the e_i equal 2. Hence the canonical sheaf on $\widetilde{X}$ is isomorphic to $\pi^*\mathcal{O}(m)$ and the proof is finished by Lemma 3.3.10. □

In Corollary 3.3.40 we found an isomorphism between $\omega_{\widetilde{X}}$ and $\pi^*\mathcal{O}(m)$ for some integer m, as the exceptional curves E_i did not contribute to the canonical class. This enabled us to immediately deduce the Kodaira dimension from m. If the E_i do contribute to the canonical class, then we can assume to know more about them in order to get the same result, an isomorphism between $\omega_{\widetilde{X}}$ and $\pi^*\mathcal{O}(m')$ for some integer m'. We will give two versions based on different assumptions.

Corollary 3.3.41 *Let again* X *be as above, a complete intersection in* $\mathbb{P}^n_K$ *of hypersurfaces of degree* $d_1, \ldots, d_{n-2}$ *with isolated singular points* Q_i *that are analytically isomorphic to the vertex of the cone over a degree* e_i *nonsingular curve* $C_i \subset \mathbb{P}^2$. *In addition, assume that for the blow-up* $\pi\colon \widetilde{X} \to X$ *of* X *at the points* Q_i *the exceptional fibres* $E_i = \pi^{-1}(Q_i)$ *are as a divisor on* $\widetilde{X}$ *all linearly equivalent with* ρ^*H *for some (any) hyperplane* H. *Set* $m' = -n-1+\sum d_j + \sum(2-e_i)$, *Then* $m'\rho^*H$ *is a canonical divisor on* $\widetilde{X}$ *and the Kodaira dimension of* X *equals*

$$\kappa(X) = \begin{cases} -1 & \textit{if } m' < 0, \\ 0 & \textit{if } m' = 0, \\ 2 & \textit{if } m' > 0. \end{cases}$$

Proof. From Proposition 3.3.35 we know that a canonical divisor is given by $m\rho^*H + \sum_{i=1}^{t}(2-e_j)E_j$ with $m = -n-1+\sum d_j$. Since E_j is linearly equivalent with ρ^*H we find that $m'\rho^*H$ is a canonical divisor as well. The proof is again finished by Lemma 3.3.10. □

Corollary 3.3.42 *Let* X, $\widetilde{X}$ *and the* E_i *be as in Corollary 3.3.41, a complete intersection in* $\mathbb{P}^n_K$ *of hypersurfaces of degree* $d_1, \ldots, d_{n-2}$ *with isolated singular points* Q_i *that are analytically isomorphic to the vertex of the cone over a degree* e_i *nonsingular curve* $C_i \subset \mathbb{P}^2$. *In addition, consider the divisor* $\sum_i (2-e_i)E_i$ *on* $\widetilde{X}$ *and assume that it is linearly equivalent with the divisor* $l\rho^*H$ *for some (any) hyperplane* H *and an integer* l. *Set* $m' = m + l$. *Then* $m'\rho^*H$ *is a canonical divisor on* $\widetilde{X}$ *and the Kodaira dimension of* X *equals*

$$\kappa(X) = \begin{cases} -1 & \textit{if } m' < 0, \\ 0 & \textit{if } m' = 0, \\ 2 & \textit{if } m' > 0. \end{cases}$$

Proof. From Proposition 3.3.35 we know that a canonical divisor is given by $m\rho^*H + \sum_{i=1}^{t}(2-e_j)E_j$ with $m = -n-1+\sum d_j$. Since $\sum_i (2-e_i)E_j$ is linearly equivalent with $l\rho^*H$ we find that $m'\rho^*H$ is a canonical divisor as well. The proof is again finished by Lemma 3.3.10. □

Example 3.3.43 Let $C \subset \mathbb{P}^2_K$ be a nonsingular curve of degree e and $X \subset \mathbb{P}^3_K$ the cone over C with vertex P. Let $\pi\colon \widetilde{X} \to X$ be the blow-up of X in the point P. Let $H \subset \mathbb{P}^3$ be a hyperplane not containing P, write $C' = \pi^*H$ and let $E = \pi^{-1}(P)$. Then Proposition 3.3.35 says that a canonical divisor on $\widetilde{X}$ is given by $(e-4)C' + (2-e)E$. This is exactly what we have seen in Example 3.3.30. Since E is linearly equivalent with C', a canonical divisor is given by $-2C' = -2\pi^*H$. Note that this means that the canonical sheaf on $X^{\mathrm{reg}} = X - \{P\}$ is pseudo-ample for $e \geq 5$, while the canonical sheaf on $\widetilde{X}$ has no global sections at all, so still $\kappa(X) = -1$.

Remark 3.3.44 We computed the Kodaira dimension of the singular variety X as that of $\widetilde{X}$. One might hope for a definition of Kodaira dimension in terms of X itself, without having to pass to a birationally equivalent nonsingular variety. In the special case that X contains only ordinary double points as singular points we have seen that the canonical divisor on $U = X^{\text{reg}}$ is pseudo-ample if and only if the canonical divisor on $\widetilde{X}$ is pseudo-ample. One might hope that this is the case under weaker assumptions, such as normality of X. However, we have seen in Example 3.3.43 that only normality is not enough to assume, even when X is a complete intersection.

On a nonsingular variety X the canonical sheaf ω_X is isomorphic to the so-called dualizing sheaf ω_X°, see [47, III.7]. Hence on these nonsingular varieties we can define the Kodaira dimension as the divisor dimension of the dualizing divisor associated to the dualizing sheaf. One might therefore hope that this does generalize to singular varieties. This would require that $\kappa(\omega_X^\circ, X)$ is a birational invariant, for so is the Kodaira dimension. However, on complete intersections X the dualizing sheaf restricts to the canonical sheaf on the regular locus X^{reg}, i.e., $\omega_X^\circ|_{X^{\text{reg}}} \cong \omega_{X^{\text{reg}}}$. For locally complete intersections this follows from Proposition 3.1.27 and Theorem III.7.11 in [47]. Taking X to be the cone of Example 3.3.43 which is regular in codimension one, it follows that the dualizing sheaf ω_X° on the cone X is isomorphic to $\mathcal{O}_X(e-4)$, this is pseudo-ample for $e \geq 5$, yet we have seen that the canonical, whence the dualizing, sheaf on $\widetilde{X}$ is not. Hence this hope evaporates by the same example 3.3.43.

In Corollary 3.3.40 we have seen that if $\widetilde{X}$ is the blow-up of a surface X in an ordinary double point P, then the exceptional fibre above P does not contribute to the canonical class of $\widetilde{X}$. This is in general true for a larger class of singular points, the so-called *A-D-E* singularities. For a proof see [35, Prop.III.3.4]. Together with Corollary 3.3.42 this implies the following Theorem.

Theorem 3.3.45 *Let $j\colon X \hookrightarrow \mathbb{P}^n_k$ be a surface which is a complete intersection in $\mathbb{P}^n_K$, say of hypersurfaces of degree $d_1, \ldots, d_{n-2}$. Suppose that X is regular except maybe for some isolated singular points $Q_1, \ldots, Q_t$. Furthermore, suppose that Q_i is either an A-D-E singularity, in which case we set $e_i = 2$, or that Q_i is analytically isomorphic to the vertex of the cone over a degree e_i nonsingular curve $C_i \subset \mathbb{P}^2$. Let $\pi\colon \widetilde{X} \to X$ be the blow-up of X at the singular points Q_i and let E_i be the exceptional divisor above Q_i. Consider the divisor $\sum_i (2-e_i)E_i$ on $\widetilde{X}$ and assume that it is linearly equivalent with the divisor $l\rho^*H$ for some (any) hyperplane H and an integer l, where $\rho = j \circ \pi$. Set $m' = m + l$, then the Kodaira dimension of X equals*

$$\kappa(X) = \begin{cases} -1 & \text{if } m' < 0, \\ 0 & \text{if } m' = 0, \\ 2 & \text{if } m' > 0. \end{cases}$$

3.4 Conjectures about surfaces of general type

S. Lang has made some conjectures about rational points on varieties of general type. For a more complete discussion see [51], [61] and [41]. Following an article of L. Caporaso, J. Harris and B. Mazur [39], we will call them the weak, the strong and the geometric Lang Conjecture, respectively. Note that the field K is no longer algebraically closed. The notions Υ and $\widetilde{\Upsilon}$ are again only used for the varieties defined over $\mathbb{Q}$.

Conjecture 3.4.1 (Weak Lang Conjecture) *If X is a variety of general type defined over a number field K, then the set $X(K)$ of K-rational points of X is not Zariski dense.*

Conjecture 3.4.2 (Strong Lang Conjecture) *Let X be any variety of general type, defined over a number field K. There exists a proper closed subvariety $\Xi \subset X$ such that for any number field L containing K, the set of L-rational points of X lying outside of Ξ is finite.*

Conjecture 3.4.3 (Geometric Lang Conjecture) *If X is any variety of general type, the union of all irreducible, positive-dimensional subvarieties of X not of general type is a proper, closed subvariety Ξ.*

Note that the Weak Lang Conjecture would imply that if X is a surface of general type over $\mathbb{Q}$, then there are finitely many curves of genus ≤ 1 on X such that there are only finitely many $\mathbb{Q}$-rational points on X that are not on any of these curves.

Note also that the Strong Lang Conjecture is implied by the Weak together with the Geometric Lang Conjecture. The Strong Lang Conjecture is also implied by a much more general conjecture of P. Vojta, see [61, Conj.3.4.3] and [41, p.1–11]. The Strong Lang Conjecture has been proven for arbitrary subvarieties of abelian varieties by Faltings [45]. The Geometric Lang Conjecture has been proved for surfaces satisfying some inequality concerning Chern numbers. This is stated more precise in Proposition 3.4.6.

Definition 3.4.4 *For a nonsingular algebraic surface X the Chern classes c_1 and c_2 are defined as follows. We have*

$$c_1(X) = -K_X \qquad \text{and} \qquad c_2(X) = 12(1+p_a) - K_X^2,$$

where K_X is a canonical divisor on X, the number K_X^2 is the self intersection number of K_X and $p_a = p_a(X)$ is the arithmetic genus of X.

Remark 3.4.5 Actually, the definition of the Chern classes is usually different, but it coincides with this definition, see [47, App.A, exa.4.1.2]. Note that from the Noether formula 3.1.52 we find that in the case of nonsingular projective surfaces over $\mathbb{C}$ we have $c_2(X) = \chi_{\text{top}}(X(\mathbb{C}))$.

Proposition 3.4.6 *Let X be a smooth minimal projective surface of general type. Suppose that for the Chern classes c_1 and c_2 we have $c_1^2 > c_2$, then X contains only finitely many curves of genus 0 or 1.*

Proof. See Bogomolov [40, Thm.0.4]. □

We have not proven that $\widetilde{\Upsilon}$ is a minimal surface. However, if we had, we still would not be able to use Proposition 3.4.6 to prove that the surface $\widetilde{\Upsilon}$ contains only finitely many curves of genus 0 or 1, for we have $c_1(\widetilde{\Upsilon})^2 < c_2(\widetilde{\Upsilon})$ by the following proposition.

Proposition 3.4.7 *For the surface $\widetilde{\Upsilon}$ we have $c_1(\widetilde{\Upsilon})^2 = 16$ and $c_2(\widetilde{\Upsilon}) = 80$.*

Proof. From Corollary 3.3.33 we know that $K_{\widetilde{\Upsilon}}^2 = 16$ for any canonical divisor $K_{\widetilde{\Upsilon}}$ on $\widetilde{\Upsilon}$. Hence we have $c_1^2 = K_{\widetilde{\Upsilon}}^2 = 16$. We have just seen that $c_2(\widetilde{\Upsilon}) = \chi_{\text{top}}(\widetilde{\Upsilon}(\mathbb{C}))$, so from Corollary 3.3.34 we find that $c_2(\widetilde{\Upsilon}) = 80$. □

The surface $\Upsilon \otimes \overline{\mathbb{Q}}$ does contain several curves of genus ≤ 1, corresponding to trivial perfect cuboids. The hyperplane $A = 0$ intersects Υ in 8 rational curves of degree 2, all in the same orbit under G and all defined over $\mathbb{Q}$. The hyperplane $X = 0$ intersects $\Upsilon \otimes \overline{\mathbb{Q}}$ in 4 elliptic curves of degree 4, all defined over $\mathbb{Q}(i)$ and in the same orbit. The hyperplane $A = B$ intersects $\Upsilon \otimes \overline{\mathbb{Q}}$ also in 4 elliptic curves of degree 4, all in the same orbit and defined over $\mathbb{Q}(\sqrt{2})$. Hence none of these elliptic curves contains any $\mathbb{Q}$-rational point. We now have three orbits of curves.

$$\left\{\begin{array}{ll} ABC = 0, & \text{24 degree 2 curves defined over } \mathbb{Q} \text{ and isomorphic to } \mathbb{P}^1, \\ U = 0, & \text{8 degree 2 curves defined over } \mathbb{Q}(i) \text{ and isomorphic to } \mathbb{P}^1. \end{array}\right.$$
$$\left\{\; XYZ = 0, \quad \text{12 degree 4 elliptic curves defined over } \mathbb{Q}(i).\right.$$
$$\left\{\begin{array}{ll} (A^2-B^2)(B^2-C^2)(C^2-A^2) = 0, & \text{24 degree 4 elliptic curves defined over } \mathbb{Q}(\sqrt{2}), \\ (A^2+U^2)(B^2+U^2)(C^2+U^2) = 0, & \text{24 degree 4 elliptic curves defined over } \mathbb{Q}(\sqrt{2}, i). \end{array}\right. \tag{36}$$

Representatives of these three orbits are

$$\begin{array}{lll}
A = 0, & A = m^2 - n^2, & A = 2mn, \\
B = m^2 - n^2, & B = 2imn, & B = 2mn, \\
C = 2mn, & C = 2mn, & C = m^2 - n^2, \\
X = m^2 + n^2, & X = 0, & X = m^2 + n^2, \\
Y = 2mn, & Y = m^2 + n^2, & Y = m^2 + n^2, \\
Z = m^2 - n^2, & Z^2 = m^4 - 6m^2n^2 + m^4, & Z = 2\sqrt{2}mn, \\
U = m^2 + n^2. & U = m^2 - n^2. & U^2 = m^4 + 6m^2n^2 + n^4.
\end{array} \tag{37}$$

Proposition 3.4.8 *There are infinitely many cuboids, pairwise not similar, with positive sides that are "perfect" over* $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

Proof. Let E denote most right elliptic curve of (37) on Υ. It contains the point

$$P = [2\sqrt{6} : 2\sqrt{6} : 1 : 5 : 5 : 4\sqrt{3} : 7]$$

corresponding with $(m, n) = (\sqrt{3}, \sqrt{2})$. The point P has infinite order, all its multiples are points over $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. □

Example 3.4.9 Let P be as in the proof of Proposition 3.4.8. Then up to torsion the point $4P$ corresponds with $(m, n) = (1382351, 174860\sqrt{6})$ giving rise to the cuboid

$$\begin{array}{rr}
A = & 483435791720\sqrt{6} \\
B = & 483435791720\sqrt{6} \\
C = & 1727438169601 \\
X = & 2094350404801 \\
Y = & 2094350404801 \\
Z = & 966871583440\sqrt{3} \\
U = & 2405943600001
\end{array}$$

Remark 3.4.10 Similarly, we find that if L is a field over which the elliptic curve $C\colon y^2 = x^4 + 6x^2 + 1$ has positive rank, then there are infinitely many cuboids that are "perfect" over $L(\sqrt{2})$. Note that E is isomorphic over $\mathbb{Q}$ to the elliptic curve $E\colon y^2 = x^3 - x$. This curve has rank 0 over $\mathbb{Q}(\sqrt{2}, i)$.

Proposition 3.4.11 *There are no straight lines on* $\Upsilon \otimes \overline{\mathbb{Q}}$.

Proof. Suppose that L were a line on $\Upsilon \otimes \overline{\mathbb{Q}}$ and consider the hyperplane $H \subset \mathbb{P}^6$ given by $Y - Z = 0$. Then by Bézout L would either lie in H, or it would have exactly one point in common with H. In the first case we find that the line would also be in the intersection of Υ with one of the hyperplanes $B \pm C$. We have already seen that these intersections only contain elliptic curves. Hence the line L intersects H in exactly one point, say P.

Since L is on Υ, we find from $Z^2 - Y^2 = B^2 - C^2$ that $B(P) = \pm C(P)$ on L. Hence either $B + C$ or $B - C$ has a single zero on L at P. It follows that either the function $(Z - Y)/(B + C)$ or $(Z - Y)/(B - C)$ has neither zeros nor poles on L. By applying ι_C to L if necessary, we may assume it is the former, which is apparantly a nonzero constant on L, say $\lambda \in \overline{\mathbb{Q}}^*$. That means that L is one component of the intersection of $\Upsilon \otimes \overline{\mathbb{Q}}$ with the hyperplane H_λ given by $f_\lambda = 0$ with $f_\lambda = Z - Y - \lambda(B + C)$. The radical ideal of the intersection $\Upsilon \cap H_\lambda$ is given by

$$I(\lambda) = \left(f_\lambda, A^2 + B^2 - Z^2, A^2 + C^2 - Y^2, B^2 + C^2 - X^2, A^2 + B^2 + C^2 - U^2\right) = I_1 \cap I_2(\lambda),$$

where I_1 and $I_2(\lambda)$ are given by

$$\begin{aligned}
I_1 &= \left(B + C, Z - Y, 2B^2 - X^2, A^2 + B^2 - Y^2, A^2 + X^2 - U^2\right), \\
I_2(\lambda) &= \left(f_\lambda, g_\lambda, B^2 + C^2 - X^2, A^2 + X^2 - U^2, 4A^2 - (\lambda - \lambda^{-1})^2 X^2 - 2(\lambda^2 - \lambda^{-2})BC\right),
\end{aligned}$$

with $g_\lambda = 2\lambda Y + (\lambda^2-1)B + (\lambda^2+1)C$. Over $\mathbb{Q}(\sqrt{2})$ the component given by I_1 falls out in two elliptic curves of degree 4, so the line L is a component of the zeroset $V(\lambda)$ of $I_2(\lambda)$. Since $V(\lambda)$ should have degree 8 in total by Bézout, there are more components and hence $I_2(\lambda)$ is not prime. We will show that $I_2(\lambda)$ is prime for $\lambda \notin \{0, \pm 1, \pm i\}$. Therefore we fix λ such that $\lambda^4 \neq 0, 1$. Since f_λ and g_λ are linear, we get an isomorphism

$$\overline{\mathbb{Q}}[A,B,C,X,Y,Z,U]/I_2(\lambda) \cong \overline{\mathbb{Q}}[A,B,C,X,U]/I_3(\lambda),$$

with

$$I_3(\lambda) = \left(B^2+C^2-X^2, A^2+X^2-U^2, 4A^2-(\lambda-\lambda^{-1})^2X^2-2(\lambda^2-\lambda^{-2})BC\right).$$

If we put

$$\begin{aligned} a_1 &= (B+iC)(B-iC)\,, \\ a_2 &= \frac{\left(\lambda^2-1\right)^2}{4\lambda^2}\left(B+\frac{\lambda+1}{\lambda-1}C\right)\left(B+\frac{\lambda-1}{\lambda+1}C\right), \\ a_3 &= \frac{\left(\lambda^2+1\right)^2}{4\lambda^2}\left(B+\frac{\lambda+i}{\lambda-i}C\right)\left(B+\frac{\lambda-i}{\lambda+i}C\right), \end{aligned} \tag{38}$$

then we can write

$$I_3(\lambda) = \left(X^2-a_1, A^2-a_2, U^2-a_3\right).$$

If we view $\overline{\mathbb{Q}}[A,B,C,X,U]$ as a polynomial ring in three variables A, X, U over the unique factorization domain $\overline{\mathbb{Q}}[B,C]$ with fraction field $L = \overline{\mathbb{Q}}(B,C)$, then we find that $\overline{\mathbb{Q}}[A,B,C,X,U]/I_3(\lambda)$ can be embedded in $L[A,X,U]/I_3(\lambda)$, so in order to show that the former is an integral domain it suffices to show that the latter is a field. Define $L_0 = L$ and

$$\begin{aligned} L_1 &= L[X]/(X^2-a_1), \\ L_2 &= L_1[A]/(A^2-a_2), \\ L_3 &= L_2[U]/(U^2-a_3). \end{aligned}$$

For $1 \leq j \leq 3$ the ring L_j is a field if L_{j-1} is a field and $a_j \notin L_{j-1}$. Using Kummer theory we find just as in the proof of Lemma 3.2.1, that $L_3 \cong L[A,X,U]/I_3(\lambda)$ is field if the sequence

$$L_0^{*2} \subset L_0^{*2}\cdot\langle a_1\rangle \subset L_0^{*2}\cdot\langle a_1,a_2\rangle \subset L_0^{*2}\cdot\langle a_1,a_2,a_3\rangle$$

of subgroups of L_0^* is strictly increasing. This follows also just as in the proof of Lemma 3.2.1 from the fact that $\overline{\mathbb{Q}}[B,C]$ is a unique factorization domain and if $\lambda \notin \{0,\pm 1,\pm i\}$, then all six linear, whence irreducible, factors in (38) are different.

We conclude that $\lambda \in \{0, \pm i, \pm 1\}$, but it is easily checked that for each of these 5 possibilities $V(\lambda)$ consists of components that we have already seen in (36), whence it does not contain a line. $\square$

4 A surface and its Néron-Severi group

So far we have found only trivial rational points on the surface Υ, i.e., rational points with $ABC = 0$. Finding non-trivial rational points on Υ appears to be rather difficult, if they exist at all. One way to search for these points is to consider the surface Υ/H for some subgroup $H \subset \mathrm{Aut}(\Upsilon)$. For every rational point found on this surface, we could check if it lifts to a rational point on Υ. The problem is now to find as many rational points on Υ/H as possible. In this chapter we will take H to be the group of order 2 generated by ι_Z. This gives a surface V and in order to find lots of rational points on V, we will try to find as many rational curves on V as possible.

4.1 A surface describing face cuboids

Definition 4.1.1 *Let V be the surface $V = \Upsilon/\langle\iota_Z\rangle$ defined over $\mathbb{Q}$ and let $\theta\colon \Upsilon \to V$ be the corresponding map.*

The surface V can be given as a surface in $\mathbb{P}^5_{\mathbb{Q}}$ by the equations

$$\begin{aligned} A^2 + C^2 - Y^2 &= 0,\\ B^2 + C^2 - X^2 &= 0,\\ A^2 + X^2 - U^2 &= 0. \end{aligned} \tag{39}$$

The map θ is then given by projection along the Z-axis. Consider the graded homogeneous coordinate ring $R = \overline{\mathbb{Q}}[A, B, C, X, Y, U]$ of $\mathbb{P}^5_{\overline{\mathbb{Q}}}$ and let $I \subset R$ be the ideal generated by the polynomials in (39). Similarly to Lemma 3.2.1 it follows that the ideal I is prime, actually it follows already from the proof of Lemma 3.2.1. Therefore V is geometrically integral and the radical ideal $I_V = I \subset \mathbb{Q}[A, B, C, X, Y, U]$ corresponding to V is prime and generated by the polynomials of (39), whence V is a complete intersection. It has degree 8 by Bézout's Theorem.

Let G_1 be the subgroup of $G \subset \mathrm{Aut}(\Upsilon \otimes \overline{\mathbb{Q}})$ consisting of all elements that commute with ι_Z. The group G_1 is commutative and generated by σ, ρ and the ι_t, where t is any of the 7 coordinates of $\mathbb{P}^6$. The group G_1 induces a group $G_V \cong G_1/\langle\iota_Z\rangle$ of automorphisms of $V \otimes \overline{\mathbb{Q}}$ of order $2^7 = 128$.

It turns out that of the 48 singular points of $\Upsilon \otimes \overline{\mathbb{Q}}$, there are 16 with $Z = 0$. These are ramification points of θ above nonsingular points of $V \otimes \overline{\mathbb{Q}}$. The other 32 singular points of $\Upsilon \otimes \overline{\mathbb{Q}}$ lie in pairs above 16 singular points of $V \otimes \overline{\mathbb{Q}}$. The group G_V acts transitively on these singular points of $V \otimes \overline{\mathbb{Q}}$. A representative of their orbit is $Q\colon [A : B : C : X : Y : U] = [1 : 0 : 0 : 0 : 1 : 1]$. Computations similar to those of Proposition 3.2.2 or computations using a Jacobian show that these 16 points are the only singular points of $V \otimes \overline{\mathbb{Q}}$. As they are isolated singular points, it follows from Proposition 3.2.17 that $V \otimes \overline{\mathbb{Q}}$, whence V, is normal. Other computations, similar to those of Lemma 3.2.9 show that Q is an ordinary double point, whence all singular points of $V \otimes \overline{\mathbb{Q}}$ are. This implies that if we blow up V at the 16 singular points, then we get a nonsingular surface and the exceptional fibre above each singular point is isomorphic to $\mathbb{P}^1$.

Definition 4.1.2 *Let $\pi\colon \widetilde{V} \to V$ be the blow-up of V at its 16 singular points.*

Remark 4.1.3 Although not all 16 singular points are defined over $\mathbb{Q}$, the singular locus is, whence so is $\widetilde{V}$.

We can apply Corollary 3.3.41 to the surface $V \otimes \overline{\mathbb{Q}}$ to get the canonical sheaf on $\widetilde{V} \otimes \overline{\mathbb{Q}}$ and the Kodaira dimension of V. With $n = 5$, $d_1 = d_2 = d_3 = 2$ and $e_1 = \cdots = e_{16} = 2$ we find $m' = 0$, so the zero divisor on $\widetilde{V} \otimes \overline{\mathbb{Q}}$ is a canonical divisor and the canonical sheaf on $\widetilde{V} \otimes \overline{\mathbb{Q}}$ is isomorphic to $\mathcal{O}_{\widetilde{V}\otimes\overline{\mathbb{Q}}}$, the structure sheaf of $\widetilde{V} \otimes \overline{\mathbb{Q}}$. Hence we find that the Kodaira dimension of V is 0 and the geometrical genus equals $p_g(V) = p_g(\widetilde{V} \otimes \otimes \overline{\mathbb{Q}}) = p_g(\widetilde{V}) = \dim H^0(\widetilde{V}, \mathcal{O}_{\widetilde{V}}) = 1$ since $\widetilde{V}$ is projective.

We can compute the Hilbert polynomial P_V of V as we did in Lemma 3.2.18 for Υ. We find that

$$P_V(n) = \sum_{k=0}^{3} \binom{3}{k} \binom{n-k+2}{2} = 4n^2 + 2,$$

so the arithmetic genus of V equals $p_a(V) = P_V(0) - 1 = 1$. Since the arithmetic genus is invariant under monoidal transformations we also find $p_a(\widetilde{V}) = 1$. For the irregularity q we then get $q = p_g(\widetilde{V}) - p_a(\widetilde{V}) = 1 - 1 = 0$. Together with the fact that the zero divisor $K_{\widetilde{V}} = 0$ is a canonical divisor, this implies that $\widetilde{V}$ is a K3 surface.

Write $h^{p,q}$ and b_k for the Hodge and betti numbers $h^{p,q}(\widetilde{V} \otimes \mathbb{C})$ and $b_k(\widetilde{V} \otimes \mathbb{C})$ of $\widetilde{V} \otimes \mathbb{C}$ respectively. Using Proposition 3.1.47 and Poincaré duality 3.1.48 we find that $h^{1,0} = h^{0,1} = h^{2,1} = h^{1,2} = q = 0$, that $h^{2,0} = h^{0,2} = p_g = 1$ and that $h^{0,0} = h^{2,2} = 1$. We have $b_0 = b_4 = 1$, $b_1 = b_3 = 0$ and $b_2 = 2 + h^{1,1}$. Using the fact that $K_{\widetilde{V}}^2 = 0$ we find from the Noether formula 3.1.52 that $h^{1,1} = 20$. All the Hodge numbers are hence given by

$$\begin{array}{|c|c|c|}\hline h^{0,2} & h^{1,2} & h^{2,2} \\ \hline h^{0,1} & h^{1,1} & h^{2,1} \\ \hline h^{0,0} & h^{1,0} & h^{2,0} \\ \hline \end{array} \quad = \quad \begin{array}{|c|c|c|}\hline 1 & 0 & 1 \\ \hline 0 & 20 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array}.$$

The surface V is also analysed by F. Beukers and B. van Geemen in [1], see section 2. We will recapitulate some of their results.

Let $W \subset \mathbb{A}^3$ be the surface given by

$$z^2 = \left(p^4 + q^2\right)\left(q^2 + 1^2\right),$$

then V and W are birationally equivalent by the following rational maps.

$$\begin{cases} p = \frac{B+X}{C}, \\ q = \frac{B+X}{A+Y}, \\ z = \frac{2U(B+X)^2}{C^2(A+Y)}. \end{cases} \qquad \text{and} \qquad \begin{cases} A = p^2 - q^2, \\ B = (p^2-1)q, \\ C = 2pq, \\ X = (p^2+1)q, \\ Y = p^2 + q^2, \\ U = z. \end{cases}$$

Let E be the elliptic curve given by

$$E: \qquad y^2 z = x^3 - 4xz^2,$$

with the 2-torsion point $T : [x : y : z] = [0 : 0 : 1]$. Then $E \times E$ is an abelian surface with automorphisms $\iota\colon (P, Q) \mapsto (-P, -Q)$ and $\gamma\colon (P, Q) \mapsto (P + T, Q + T)$, both of order 2. The map

$$\Phi\colon E \times E \to V$$

given by

$$\begin{cases} A = y_1^2 y_2^2 - 16x_1^2 x_2^2, \\ B = 4(y_1^2 x_2^2 - y_2^2 x_1^2), \\ C = 8x_1 x_2 y_1 y_2, \\ X = 4(y_1^2 x_2^2 + y_2^2 x_1^2), \\ Y = y_1^2 y_2^2 + 16x_1^2 x_2^2, \\ U = (y_1^2 + 8x_1 z_1)(y_2^2 + 8x_2 z_2) \end{cases}$$

is a morphism that factors through $(E \times E)/\langle \iota, \gamma \rangle$, inducing an isomorphism between V and $(E \times E)/\langle \iota, \gamma \rangle$.

Let $\widetilde{\Phi}$ be the morphism $\widetilde{\Phi} = \Phi \circ \alpha$, where α is the automorphism $\alpha\colon (P, Q) \mapsto (P + Q, Q)$ of $E \times E$, i.e.,

$$\widetilde{\Phi}\colon E \times E \to V\colon (P, Q) \mapsto \Phi(P + Q, Q).$$

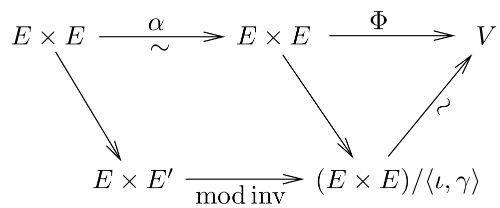
Let τ be the automorphism (involution) of E that sends P to $P + T$ and let E' denote the elliptic curve $E' = E/\langle \tau \rangle$. Then E' can be given by

$$E'\colon \quad y^2 z = x^3 + xz^2.$$

Let τ' be the automorphism $\tau' = \mathrm{id} \times \tau$ of $E \times E$, then $(E \times E)/\langle \tau' \rangle \cong E \times E'$. Note that we have $\alpha^{-1} \langle \iota, \gamma \rangle \alpha = \langle \tau', \iota \rangle$, whence we have isomorphisms

$$V \cong (E \times E)/\langle \iota, \gamma \rangle \cong (E \times E)/\langle \tau', \iota \rangle \cong (E \times E')/\langle \mathrm{inv} \rangle,$$

where inv is the automorphism $\mathrm{inv}\colon (P, Q) \mapsto (-P, -Q)$ of $E \times E'$.



This implies that V is the Kummer surface of the abelian surface $E \times E'$.

4.2 Generators of the Néron-Severi group

In this section we will give generators for the Néron-Severi group of $\widetilde{V} \otimes \mathbb{C}$. This is the free group generated by curves on $\widetilde{V} \otimes \mathbb{C}$ modulo algebraic equivalence. Therefore, in order to find generators, we'd better have some curves at our disposal.

The hyperplane $C = 0$ intersects V in 4 conics given by

$$C = 0, \qquad Y = \pm A, \qquad X = \pm B, \qquad A^2 + B^2 = U^2.$$

We will denote these by $D_{Cj_1j_2}$, where $j_1 = \operatorname{sign} Y/A$ and $j_2 = \operatorname{sign} X/B$, whence D_{C+-} for example denotes the curve given by $C = 0$, $Y = A$, $X = -B$ and $A^2 + B^2 = U^2$. The hyperplane $A = 0$ intersects V in 4 conics as well, given by

$$A = 0, \qquad Y = \pm C, \qquad X = \pm U, \qquad B^2 + C^2 = X^2.$$

We will denote these by $D_{Aj_1j_2}$, with $j_1 = \operatorname{sign} Y/C$ and $j_2 = \operatorname{sign} X/U$. Similarly we find 4 conics $D_{Bj_1j_2}$ with $B = 0$ and $j_1 = \operatorname{sign} X/C$ and $j_2 = \operatorname{sign} Y/U$. The hyperplane $Y = 0$ intersects V in an algebraic set that over $\overline{\mathbb{Q}}$ falls out in 4 conics, namely

$$iA = \pm C, \qquad B = \pm U, \qquad Y = 0, \qquad B^2 + C^2 = X^2$$

where i is fixed and $i^2 = -1$. These conics will be denoted by $D_{Yj_1j_2}$ with $j_1 = \operatorname{sign} iA/C$ and $j_2 = \operatorname{sign} B/U$. Analogously, we find 4 conics $D_{Xj_1j_2}$ with $j_1 = \operatorname{sign} iB/C$ and $j_2 = \operatorname{sign} A/U$ in the intersection of V with the hyperplane $X = 0$. Finally, the hyperplane given by $U = 0$ intersects $V \otimes \overline{\mathbb{Q}}$ in the 4 conics with

$$iX = \pm A, \qquad iY = \pm B, \qquad B^2 + C^2 = X^2, \qquad U = 0.$$

These 4 conics will be denoted $D_{Uj_1j_2}$ with $j_1 = \operatorname{sign} iX/A$ and $j_2 = \operatorname{sign} iY/B$. Each of these 24 conics contains exactly 4 singular points and each singular point is contained in 6 of the conics. It turns out that there are no straight lines on $V \otimes \overline{\mathbb{Q}}$.

Proposition 4.2.1 *There are no straight lines on $V \otimes \overline{\mathbb{Q}}$.*

Proof. Suppose that L were a line on V. Then L can be parametrised by linear homogeneous polynomials in s and t, i.e., there are homogeneous linear polynomials $f_A, f_B, f_C, f_X, f_Y, f_U \in \overline{\mathbb{Q}}[s,t]$ such that

$$\begin{aligned} f_A^2 + f_C^2 &= f_Y^2, \\ f_B^2 + f_C^2 &= f_X^2, \\ f_A^2 + f_X^2 &= f_U^2. \end{aligned}$$

Suppose $f_A = a_1 s + a_2 t$, $f_C = c_1 s + c_2 t$ and $f_Y = y_1 s + y_2 t$. Then from the first equation we find $y_1^2 = a_1^2 + c_1^2$, $y_2^2 = a_2^2 + c_2^2$ and $y_1 y_2 = a_1 a_2 + c_1 c_2$, whence

$$(a_1^2 + c_1^2)(a_2^2 + c_2^2) = y_1^2 y_2^2 = (a_1 a_2 + c_1 c_2)^2,$$

which rewrites to $(a_1 c_2 - a_2 c_1)^2 = 0$, whence $a_1 c_2 = a_2 c_1$. If $a_1 \neq 0$, then $f_C = \frac{c_1}{a_1} f_A$. If $a_2 \neq 0$, then $f_C = \frac{c_2}{a_2} f_A$ and if $a_1 = a_2 = 0$, then $f_A = 0$. In all cases there exists a linear form $g \in \overline{\mathbb{Q}}[s,t]$ and constants $a, c \in \overline{\mathbb{Q}}$ such that $f_A = ag$ and $f_C = cg$. Then $f_Y = yg$ for some constant y with $y^2 = a^2 + c^2$. Together with the other two equations we find similarly that there is a linear form $g' \in \overline{\mathbb{Q}}[s,t]$ with constants $a, b, c, x, y, u \in \overline{\mathbb{Q}}$ such that $f_A = ag', f_B = bg', \ldots, f_U = ug'$. This implies that L is just the point $[a : b : c : x : y : u]$, contradiction. $\square$

Remark 4.2.2 Proposition 3.4.11 follows from this proposition as every straight line on $\Upsilon \otimes \overline{\mathbb{Q}}$ would map to a straight line on $V \otimes \overline{\mathbb{Q}}$. The proof of Proposition 3.4.11 therefore seems to be more complicated than needed.

On $\widetilde{V} \otimes \overline{\mathbb{Q}}$ we have 16 more rational curves, the exceptional curves above the singular points. We name the singular points as follows.

$$\begin{array}{llll} Q_1 = [0:1:0:1:0:1], & Q_2 = \iota_U Q_1, & Q_3 = \iota_X Q_1, & Q_4 = \iota_B Q_1, \\ Q_5 = [1:0:0:0:1:1], & Q_6 = \iota_U Q_5, & Q_7 = \iota_Y Q_5, & Q_8 = \iota_A Q_5, \\ Q_9 = [0:i:1:0:1:0], & Q_{10} = \iota_B Q_9, & Q_{11} = \iota_C Q_9, & Q_{12} = \iota_Y Q_9, \\ Q_{13} = [i:0:1:1:0:0], & Q_{14} = \iota_X Q_{13}, & Q_{15} = \iota_C Q_{13}, & Q_{16} = \iota_A Q_{13}. \end{array}$$

For $j = 1, \ldots, 16$ the exceptional curve above Q_j is isomorphic to $\mathbb{P}^1$ and will be denoted E_j.

Let λ and ζ be coordinates of $\mathbb{P}^1$ and consider the rational map

$$\varphi: V \dashrightarrow \mathbb{P}^1: [A:B:C:X:Y:U] \mapsto \begin{cases} [Y - A : C] & \text{or} \\ [C : Y + A], & \end{cases}$$

which is only not well defined in the four singular points with $A = C = Y = 0$, i.e., in Q_1, Q_2, Q_3 and Q_4.

Definition 4.2.3 *Let $\mathcal{E}'_\lambda$ denote the inverse image of $[\lambda : 1]$ under φ and write $\mathcal{E}'_\infty$ for the inverse image of $[1 : 0]$, all of them not including the points Q_1, Q_2, Q_3, Q_4. Let $\mathcal{E}_\lambda$ denote the closure of $\mathcal{E}'_\lambda$ in V, i.e., $\mathcal{E}_\lambda = \mathcal{E}'_\lambda \cup \{Q_1, Q_2, Q_3, Q_4\}$ and let $\widetilde{\mathcal{E}}_\lambda$ denote the closure of $\pi^{-1}(\mathcal{E}'_\lambda)$ in $\widetilde{V}$.*

Let K by any field extension of $\mathbb{Q}$. For $\lambda \in K$ let $H_\lambda, H'_\lambda \subset \mathbb{P}^5_K$ denote the hyperplanes given by $Y - A = \lambda C$ and $\lambda(Y + A) = C$ respectively. The hyperplane H_λ cuts out on $V \otimes K$ the two conics $D_{C+\pm}$ and $\mathcal{E}_\lambda \otimes K$. After intersecting with H'_λ only $\mathcal{E}_\lambda \otimes K$ remains left. This helps us to describe $\mathcal{E}_\lambda$, which in turn will enable us to describe $\widetilde{\mathcal{E}}_\lambda$ on $\widetilde{V}$.

For $\lambda = 0 \in \mathbb{Q}$ we find that $\mathcal{E}_\lambda$ consists of the two conics $D_{C+\pm}$ and for $\lambda = \infty$ of the conics $D_{C-\pm}$. For $\lambda \neq 0, \infty$ the radical ideal of $\mathcal{E}_\lambda$ is given by

$$I_\lambda = \left(Y - A - \lambda C, 2A + (\lambda - \lambda^{-1})C, B^2 + C^2 - X^2, 4B^2 + (\lambda + \lambda^{-1})^2 C^2 - 4U^2\right). \qquad (40)$$

As the first two polynomials in (40) are linear, it follows that $\mathcal{E}_\lambda$ is isomorphic to the intersection of two quadratic surfaces in $\mathbb{P}^3_K$, namely

$$B^2 + C^2 - X^2 \qquad \text{and} \qquad 4B^2 + (\lambda + \lambda^{-1})^2 C^2 - 4U^2. \qquad (41)$$

With the distinguished point

$$\mathcal{O}(B : C : X : U) = [-1 : 0 : 1 : 1]$$

this becomes an elliptic curve over K for almost all $\lambda \in K$. Now consider $K = \overline{\mathbb{Q}}$. The curve is singular for $\lambda = 0, \pm i, \pm 1, \infty$. For these λ the intersection $\mathcal{E}_\lambda$ decomposes on $V \otimes \overline{\mathbb{Q}}$ as the sum

$$\begin{aligned}
\mathcal{E}_1 &: D_{A++} + D_{A+-},\\
\mathcal{E}_{-1} &: D_{A-+} + D_{A--},\\
\mathcal{E}_i &: D_{Y++} + D_{Y+-},\\
\mathcal{E}_{-i} &: D_{Y-+} + D_{Y--},\\
\mathcal{E}_0 &: D_{C++} + D_{C+-},\\
\mathcal{E}_\infty &: D_{C-+} + D_{C--}.
\end{aligned}$$

Proposition 4.2.4 *There is a morphism* $\widetilde{\varphi}: \widetilde{V} \to \mathbb{P}^1$ *such that for* $P \in \widetilde{V} - (E_1 \cup E_2 \cup E_3 \cup E_4)$ *we have* $\widetilde{\varphi}(P) = \varphi(\pi(P))$.

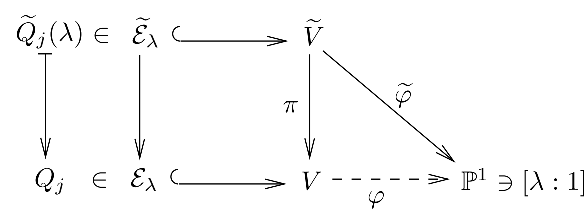
Proof. The problem is to extend $\varphi \circ \pi$ to the whole of $\widetilde{V}$. For $\lambda \neq 0, \pm 1, \pm i, \infty$ the fibre $\mathcal{E}_\lambda$ is nonsingular, in particular at Q_1, Q_2, Q_3 and Q_4. For $j = 1, 2, 3, 4$ it follows that $\widetilde{\mathcal{E}}_\lambda$ intersects E_j exactly in 1 point, which we will call $\widetilde{Q}_j(\lambda)$.

For each $\lambda = 0, \infty, \pm 1, \pm i$ both the two nonsingular conics of which $\mathcal{E}_\lambda$ consists contain exactly 2 of the 4 points Q_1, Q_2, Q_3, Q_4. The two conics intersect at 2 other points that are singular on V. Hence for $j = 1, 2, 3, 4$ again $\mathcal{E}_\lambda$ is nonsingular at Q_j, so $\widetilde{\mathcal{E}}_\lambda$ intersects E_j also for these λ exactly in 1 point, which we will call $\widetilde{Q}_j(\lambda)$ again.

Fix $j \in \{1, 2, 3, 4\}$. For different λ the curve $\mathcal{E}_\lambda$ goes through Q_j with an other direction, so the morphism

$$\mathbb{P}^1 \to E_j : \lambda \mapsto \widetilde{Q}_j(\lambda)$$

is injective. It is not constant, whence it is surjective so it is an isomorphism. This implies that we can extend the rational map $\varphi \circ \pi: \widetilde{V} \to \mathbb{P}^1$, which is a priori only defined outside E_1, E_2, E_3, E_4, to a morphism $\widetilde{\varphi}: \widetilde{V} \to \mathbb{P}^1$ by sending $\widetilde{Q}_j(\lambda)$ to $[\lambda : 1]$ for $j = 1, 2, 3, 4$.



Although we worked with an extension of $\mathbb{Q}$, the morphism $\widetilde{\varphi}$ is defined over $\mathbb{Q}$ itself. $\square$

The inverse image of $[\lambda : 1]$ under $\widetilde{\varphi}$ is exactly $\widetilde{\mathcal{E}}_\lambda$. For $\lambda \neq 0, \infty, \pm 1, \pm i$ the irreducible nonsingular curve $\mathcal{E}_\lambda \otimes \overline{\mathbb{Q}}$ on $V \otimes \overline{\mathbb{Q}}$ does not contain any singular points other than Q_1, Q_2, Q_3 and Q_4, so $\widetilde{\mathcal{E}}_\lambda \otimes \overline{\mathbb{Q}}$ does not contain any exceptional curves and is isomorphic to $\mathcal{E}_\lambda \otimes \overline{\mathbb{Q}}$. To describe $\widetilde{\mathcal{E}}_\lambda \otimes \overline{\mathbb{Q}}$ for $\lambda = 0, \infty, \pm 1, \pm i$ it is convenient to have a notation for the conics on $\widetilde{V}$ that are isomorphic under π to conics on V.

Definition 4.2.5 *Let D be a curve on V, which is nonsingular at the singular points of V. Then the closure of $\pi^{-1}(D \cap V^{\text{reg}})$ in $\widetilde{V}$ is isomorphic to D and will be denoted $\widetilde{D}$.*

Lemma 4.2.6 *Consider the morphism $\widetilde{\varphi} \otimes \overline{\mathbb{Q}}: \widetilde{V} \otimes \overline{\mathbb{Q}} \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$. The fibre of $\widetilde{\varphi} \otimes \overline{\mathbb{Q}}$ is singular at $\lambda = 0, \infty, \pm 1, \pm i$. These fibres decompose as follows.*

$$\begin{aligned}
\widetilde{\mathcal{E}}_0&: \quad \widetilde{D}_{C++} + \widetilde{D}_{C+-} + E_5 + E_6,\\
\widetilde{\mathcal{E}}_\infty&: \quad \widetilde{D}_{C-+} + \widetilde{D}_{C--} + E_7 + E_8,\\
\widetilde{\mathcal{E}}_1&: \quad \widetilde{D}_{A++} + \widetilde{D}_{A+-} + E_9 + E_{10},\\
\widetilde{\mathcal{E}}_{-1}&: \quad \widetilde{D}_{A-+} + \widetilde{D}_{A--} + E_{11} + E_{12},\\
\widetilde{\mathcal{E}}_i&: \quad \widetilde{D}_{Y++} + \widetilde{D}_{Y+-} + E_{13} + E_{14},\\
\widetilde{\mathcal{E}}_{-i}&: \quad \widetilde{D}_{Y-+} + \widetilde{D}_{Y--} + E_{15} + E_{16}.
\end{aligned}$$

For each $\lambda = 0, \infty, \pm 1, \pm i$ the two conics in the decomposition of $\widetilde{E}_\lambda \otimes \overline{\mathbb{Q}}$ do not intersect each other on $\widetilde{V}$, neither do the two exceptional curves. The conics both have exactly one (different) point in common with each of the exceptional curves.

Proof. First consider $\lambda = 0$ and the conics $D_{C+\pm}$ of which $\mathcal{E}_0$ on V consists. Apart from the points Q_1, Q_2, Q_3 and Q_4, the only singular points of V on $D_{C+\pm}$ are Q_5 and Q_6. These are the intersection points of D_{C++} and D_{C+-}. Hence the only exceptional curves in the pre-image of $\mathcal{E}_0' = \mathcal{E}_0 - \{Q_1, Q_2, Q_3, Q_4\}$ under π are the exceptional curves E_5 and E_6. It follows that $\widetilde{\mathcal{E}}_0$ on $\widetilde{V}$ consists of 4 irreducible curves, namely E_5, E_6 and the conics $\widetilde{D}_{C+\pm}$.

Similarly, the only singular points of V on $\mathcal{E}_\infty$, apart from Q_1, Q_2, Q_3 and Q_4, are Q_7 and Q_8 and these are the intersection points of the conics $D_{C-\pm}$. Then $\widetilde{\mathcal{E}}_\infty$ consists of E_7, E_8 and the conics $\widetilde{D}_{C-\pm}$ on $\widetilde{V}$. Note that $\widetilde{\mathcal{E}}_\infty$ consists of 4 irreducible curves as well.

Just as before, the only singular points of $V \otimes \overline{\mathbb{Q}}$ on $\mathcal{E}_\lambda \otimes \overline{\mathbb{Q}}$, apart from Q_1, Q_2, Q_3, Q_4, for $\lambda = 1, -1, i, -i$ are the pairs (Q_9, Q_{10}), (Q_{11}, Q_{12}), (Q_{13}, Q_{14}) and (Q_{15}, Q_{16}) respectively. The decompositions follow just as in the case of $\lambda = 0$.

In all six cases each of the two conics is nonsingular at the two intersection points Q_i and Q_j at which the conics intersect transversally. It follows that the conics do not intersect anymore on $\widetilde{V} \otimes \overline{\mathbb{Q}}$ and that they both intersect E_i and E_j in exactly one (different) point. □

The pre-image of the generic point of $\mathbb{P}^1_{\mathbb{Q}}$ under φ is the elliptic curve over the function field $\mathbb{Q}(\lambda)$ of $\mathbb{P}^1$, also denoted by $\mathcal{E}_\lambda$. It can be given in $\mathbb{P}^3_{\mathbb{Q}(\lambda)}$ as the intersection of the two quadrics given by (41) and it can be brought into Weierstrass form

$$\mathcal{C}_\lambda: \quad y^2 z = x(x + 4\lambda^2 z)(x + (\lambda^2+1)^2 z)$$

by the map

$$\begin{cases} A = 2(1-\lambda^2)yz,\\ B = (x - 2\lambda(\lambda^2+1)z)(x + 2\lambda(\lambda^2+1)z),\\ C = 4\lambda yz,\\ X = x^2 + 8\lambda^2 xz + 4\lambda^2(\lambda^2+1)^2 z^2,\\ Y = 2(\lambda^2+1)yz,\\ U = (x + 2(\lambda^2+1)z)(x + 2\lambda^2(\lambda^2+1)z) \end{cases} \quad \text{or} \quad \begin{cases} x = 4\lambda^2 C(U-B),\\ y = 8\lambda^3(U+X)(U-B),\\ z = C(X+B). \end{cases}$$

The elliptic curve $\mathcal{C}_\lambda/\mathbb{Q}(\lambda)$ has discriminant $\Delta(\lambda)$ and j-value $j(\lambda)$ given by

$$\Delta(\lambda) = 2^8 \lambda^4 (\lambda^2+1)^4 (\lambda+1)^4 (\lambda-1)^4 \qquad \text{and}$$
$$j(\lambda) = 16 \frac{(\lambda^4 + 2\lambda^3 + 2\lambda^2 - 2\lambda + 1)(\lambda^4 - 2\lambda^3 + 2\lambda^2 + 2\lambda + 1)}{\lambda^4(\lambda^2+1)^4(\lambda+1)^4(\lambda-1)^4}.$$

Lemma 4.2.7 *Let $\phi: \mathcal{S} \to \mathbb{P}^1$ be a nonsingular minimal model for $\varphi: \mathcal{E}_\lambda \to \mathbb{P}^1$. The nonsingular fibres of $\phi \otimes \overline{\mathbb{Q}}$ at $\lambda = 0, \infty, \pm 1, \pm i$ are all of type I_4.*

Proof. This follows from Tate's algorithm. For $\lambda = 0, \pm 1, \pm i$ it follows directly from the valuation of $\Delta(\lambda)$ and $j(\lambda)$. For $\lambda = \infty$ we should first do some rewriting to see that $\mathcal{E}_\lambda$ looks locally at $\lambda = \infty$ exactly the same as at $\lambda = 0$. □

Proposition 4.2.8 *The surface $\widetilde{V}$ together with the morphism $\widetilde{\varphi}$ gives a minimal nonsingular model for $\mathcal{E}_\lambda$.*

Proof. As we have seen the surface $\widetilde{V}$ is nonsingular, so it is minimal if and only if the (singular) fibres are what they should be as described by the Tate algorithm. For $\lambda \neq 0, \infty, \pm 1, \pm i$ the fibre $\widetilde{\mathcal{E}}_\lambda \otimes \overline{\mathbb{Q}}$ is indeed an elliptic curve isomorphic to $\mathcal{E}_\lambda \otimes \overline{\mathbb{Q}}$, for π induces an isomorphism outside the singular points of V. The singular fibres should all be of type I_4 and by Lemma 4.2.6 this is exactly what the singular fibres of $\widetilde{\varphi}: \widetilde{V} \to \mathbb{P}^1$ look like. □

Let $NS(\widetilde{V} \otimes \mathbb{C}, \mathbb{C})$ denote the Néron-Severi group of $\widetilde{V} \otimes \mathbb{C}$ over $\mathbb{C}$. Then $NS(\widetilde{V} \otimes \mathbb{C}, \mathbb{C})$ is a finitely generated $\mathbb{Z}$-module and it follows from Shioda [56, Cor.1.5] that

$$\operatorname{rank} NS(\widetilde{V} \otimes \mathbb{C}, \mathbb{C}) = \operatorname{rank} \mathcal{E}_\lambda(\mathbb{C}(\lambda)) + 2 + 6 \cdot (4 - 1).$$

Since the rank of the Néron-Severi group of a K3-surface cannot exceed 20 (see [35]) it follows that $\operatorname{rank} NS(\widetilde{V}, \mathbb{C}) = 20$ and $\operatorname{rank} \mathcal{E}_\lambda(\mathbb{C}(\lambda)) = 0$. This means that the Mordell-Weil group of $\mathcal{E}_\lambda(\mathbb{C}(\lambda))$ is a torsion group. Three points on $\mathcal{E}_\lambda$ are

$$\begin{aligned}
T_1: \quad [A:B:C:X:Y:U] \quad &= \left[1-\lambda^2 : 0 : 2\lambda : 2\lambda : 1+\lambda^2 : 1+\lambda^2\right], \\
T_2: \quad [A:B:C:X:Y:U] \quad &= \left[1-\lambda^2 : i(1+\lambda^2) : 2\lambda : i(1-\lambda^2) : 1+\lambda^2 : 0\right] \\
T_3: \quad [A:B:C:X:Y:U] \quad &= [0:1:0:-1:0:1],
\end{aligned}$$

corresponding with

$$\begin{aligned}
T_1 &: [x:y:z] = [2\lambda(\lambda^2+1) : 2\lambda(\lambda^2+1)(\lambda+1)^2 : 1], \\
T_2 &: [x:y:z] = [-2(\lambda^2+1) : 2i(\lambda^2+1)(\lambda^2-1) : 1], \\
T_3 &: [x:y:z] = [-4\lambda^2 : 0 : 1].
\end{aligned}$$

We will see in Proposition 4.2.12 that T_1 and T_2 generate the Mordell-Weil group over $\mathbb{C}(\lambda)$. For this proposition we need a generalization of the Theorem of Lutz and Nagell, see [57, Cor.VIII.7.2]. This can be stated much more general than the version we need, which is stated in Corollary 4.2.11.

Proposition 4.2.9 *Let R be a discrete valuation ring with quotient field K and valuation v. Let $\mathfrak{m}$ be the maximal ideal of R and $k = R/\mathfrak{m}$ its residue field. Let $m \geq 1$ be an integer relatively prime to* $\operatorname{char} k$. *Let E/K be an elliptic curve, $\widetilde{E}/k$ the reduction of E modulo $\mathfrak{m}$ and set*

$$E_1(K) = \{P \in E(K) : \widetilde{P} = \widetilde{\mathcal{O}}\}.$$

Then the subgroup $E_1(K)$ of $E(K)$ has no non-trivial points of order m.

Proof. Let $\hat{R}$ and $\hat{K}$ denote the completions with respect to v of R and K respectively. Then $E_1(K)$ is a subgroup of

$$E_1(\hat{K}) = \{P \in E(\hat{K}) : \widetilde{P} = \widetilde{\mathcal{O}}\},$$

which has no non-trivial points of order m by Proposition VII.3.1(a) of [57]. □

Remark 4.2.10 It is easily checked that the proof Proposition VII.3.1 of [57] does not use completeness. The completeness is only used for right exactness of an exact sequence of which right exactness is not used.

Corollary 4.2.11 *Let k be a field of characteristic* $\operatorname{char} k = 0$ *and let K be the field $K = k(t)$, where t is transcendental over k. Let E/K be an elliptic curve given by the Weierstrass equation*

$$E: \quad y^2 = x^3 + a_2x^2 + a_4x + a_6 =: f(x)$$

with $a_2, a_4, a_6 \in k[t]$. Let $P \in E(K)$ be a non-zero torsion point. Then

(a) $x(P), y(P) \in k[t]$ and

(b) $2P = \mathcal{O}$ or $y(P)^2 | \Delta(E) = -16(4a_2^3a_6 - a_2^2a_4^2 + 4a_4^3 + 27a_6^2 - 18a_2a_4a_6)$.

Proof. Let $P \in E(K)$ be a point of order m. Let $\mathfrak{m}$ be any nonzero prime ideal of $k[t]$ and let v be the corresponding valuation. We will show that $v(x(P)) \geq 0$. If the equation for E is not minimal with respect to v, and (x', y') are coordinates for a minimal equation, then we have

$$v(x(P)) \geq v(x'(P)) \qquad \text{and} \qquad v(y(P)) \geq v(y'(P))$$

(see [57, Prop.VII.1.3d]). Hence we may assume that the equation for E is minimal with respect to v. Let R be the localization $R = k[t]_{\mathfrak{m}}$ of $k[t]$ at $\mathfrak{m}$. Then R is a discrete valuation ring with respect to v and with quotient field K. The residue field $l = R/\mathfrak{m}R$ is a finite field extension of k. Let $\widetilde{E}/l$ be the reduction of E modulo $\mathfrak{m}$.

Suppose that we had $v(x(P)) < 0$. Then we would also have $v(y(P)) < 0$ and from minimality of the equation for E we find that P reduces to $\widetilde{P} = \widetilde{\mathcal{O}}$, whence $P \in E_1(K)$. Since $\operatorname{char} l = 0$ we can apply Proposition 4.2.9 to find that P is not a non-trivial point of order m. From this contradiction we conclude that $v(x(P)) \geq 0$. This holds for every valuation corresponding to a nonzero prime ideal of $k[t]$, so indeed we find $x[t] \in k[t]$. This immediately implies that $y(P)^2 = x(P)^3 + a_2x(P)^2 + a_4x(P) + a_6 \in k[t]$, whence $y(P) \in k[t]$, thus proving part (a).

To prove part (b), suppose that $P \in E(K)$ is a non-zero torsion point with $2P \neq \mathcal{O}$. Then from part (a) we know that $x(P), y(P), x(2P), y(2P) \in k[t]$. Let $\psi, \phi \in k[t][x]$ be given by

$$\psi = -27x^3 - 27a_2x^2 - 27a_4x + 4a_2^3 - 18a_2a_4 + 27a_6,$$
$$\phi = 3x^2 + 2a_2x - a_2^2 + 4a_4.$$

Then it is directly verified that $\psi(x)f(x) + \phi(x)f'(x)^2 = -\Delta(E)/16 \in k[t]$. Note that we also have the following equations.

$$4y(P)^2\left(x(2P) + a_2 + 2x(P)\right) = f'(x(P))^2$$
$$y(P)^2 = f(x(P))$$

The first one follows from the addition formula for $x(2P)$ and the second from the fact that P is a point on E. Multiplying the first by ϕ, the second by ψ and adding them up gives

$$y(P)^2\left(4\phi(x(2P) + a_2 + 2x(P)) + \psi\right) = -\Delta(E)/16.$$

As all these factors are contained in the unique factorization domain $k[t]$, we conclude that $y(P)^2 | \Delta(E)$. $\square$

Proposition 4.2.12 *The Mordell Weil group $\mathcal{E}_\lambda(\mathbb{C}(\lambda))$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and generated by T_1 and T_2. The group $\mathcal{E}_\lambda(\mathbb{Q}(\lambda))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ and generated by T_1 and T_3.*

Proof. Let $P \neq \mathcal{O}$ be any torsion point in $\mathcal{E}_\lambda(\mathbb{C}(\lambda))$. Then from Corollary 4.2.11 we know that $x(P), y(P) \in \mathbb{C}[\lambda]$ and that either $2P = \mathcal{O}$ or

$$y(P)^2 = x(P)\left(x(P) + 4\lambda^2\right)\left(x(P) + (\lambda^2 + 1)^2\right)$$

is a divisor of

$$\Delta(\lambda) = 2^8\lambda^4(\lambda+i)^4(\lambda-i)^4(\lambda+1)^4(\lambda-1)^4.$$

As $\mathbb{C}[\lambda]$ contains infinitely many units, this still leaves infinitely many possibilities. First note that we can narrow down the possible degrees for $x(P)$. As the degree of $f(x(P)) = y(P)^2$ as a polynomial in $\mathbb{C}[\lambda]$ should be even and at most $\deg\Delta(\lambda) = 20$, it follows from the following tabular that the degree of $x(P) \in \mathbb{C}[\lambda]$ equals $0, 2, 3, 4$ or 6.

$\deg x(P)$	$\deg f(x(P))$
≥ 7	≥ 21
6	18
5	15
4	$8 + \deg(x(P) + (\lambda^2+1)^2)$
3	10
2	$6 + \deg(x(P) + 4\lambda^2)$
1	7
0	6

We also know that $(x(P), x(P)+4\lambda^2)$ is a pair of polynomials (g, h) in $\mathbb{C}[\lambda]$ both dividing $\Delta(\lambda)$, whence with $g = cg'$ and $h = dh'$ for some constants $c, d \in \mathbb{C}$ and monic divisors g', h' of $\Delta(\lambda)$, say of degree n and m, of the form

$$g' = \lambda^*(\lambda+i)^*(\lambda-i)^*(\lambda+1)^*(\lambda-1)^* = \sum_{j=0}^{n} c_j\lambda^j,$$

$$h' = \lambda^*(\lambda+i)^*(\lambda-i)^*(\lambda+1)^*(\lambda-1)^* = \sum_{j=0}^{m} d_j\lambda^j,$$

where the stars mean any exponent from 0 to 4. Then g and h satisfy $h = g + 4\lambda^2$, whence

$$g' - c_2\lambda^2 = \alpha(h' - d_2\lambda^2) \tag{42}$$

for some $\alpha \in \mathbb{C}$. Furthermore we have $n = 0, 2, 3, 4, 6$ and

$$m = \begin{cases} n & \text{if} \quad n = 3, 4, 6, \\ 0 \text{ or } 2 & \text{if} \quad n = 2, \\ 2 & \text{if} \quad n = 0. \end{cases}$$

Hence for these pairs (m, n) we can compute all polynomials of the form

$$\lambda^*(\lambda+i)^*(\lambda-i)^*(\lambda+1)^*(\lambda-1)^*$$

of degree m and n. In this finite set of polynomials we can look for all pairs (g', h') satisfying (42). Then from the equations $g = cg'$, $h = dh'$ and $h = g + 4\lambda^2$ we can compute the constants c and d by looking at the coefficients. This gives a finite set of all possible polynomials g for $x(P)$. We let a computer compute all these possibilities and check for each possibility whether $f(x(P))$ was a square in $\mathbb{C}[\lambda]$. Out came 9 polynomials, 3 of which are 0, $-4\lambda^2$ and $-(\lambda^2+1)^2$ giving rise to 2-torsion points. The other 6 give rise to 12 points of order 4. Together with $\mathcal{O}$ this gives a total of 16 points, all contained in the group generated by T_1 and T_2 which is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

As the group $\mathcal{E}_\lambda(\mathbb{C}(\lambda))$ is not invariant under complex conjugation, the group $\mathcal{E}_\lambda(\mathbb{Q}(\lambda))$ has at least index 2 in $\mathcal{E}_\lambda(\mathbb{C}(\lambda))$, so at most order 8. The group generated by T_1 and T_3 is contained in $\mathcal{E}_\lambda(\mathbb{Q}(\lambda))$ and is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$, whence $\mathcal{E}_\lambda(\mathbb{Q}(\lambda))$ is generated by T_1 and T_3. □

Knowing that $\widetilde{\varphi}\colon \widetilde{V} \to \mathbb{P}^1$ is a minimal nonsingular model for $\mathcal{E}_\lambda$ we can find generators of $NS(\widetilde{V} \otimes \mathbb{C}, \mathbb{C})$ by the ideas of Swinnerton-Dyer [59]. These tell us that the group $NS(\widetilde{V} \otimes \mathbb{C}, \mathbb{C})$ is spanned over $\mathbb{Z}$ by

- the locus of the point $\mathcal{O}_\lambda$,
- the components of the singular fibres in the pencil $\mathcal{E}_\lambda$ and
- the loci of the generators T_1, T_2 of the group $\mathcal{E}_\lambda(\mathbb{C}(\lambda))$.

Lemma 4.2.13 *The Néron-Severi group $NS(\widetilde{V} \otimes \mathbb{C}, \mathbb{C})$ of $\widetilde{V} \otimes \mathbb{C}$ is generated by the exceptional curves $E_4, \ldots, E_{16}$ and the conics $\widetilde{D}_{B++}$, D_{U-+}, $\widetilde{D}_{A\pm\pm}$, $\widetilde{D}_{Y\pm\pm}$, $\widetilde{D}_{C\pm\pm}$.*

Proof. The locus of the point $\mathcal{O}_\lambda$ on $\widetilde{V}$ is E_4. The locus of the points T_1 and T_2 on $\widetilde{V}$ are the conics $\widetilde{D}_{B++}$ and $\widetilde{D}_{U-+}$ respectively. From the ideas of Swinnerton-Dyer it follows that we get a set of generators by adding the irreducible components of the singular fibres given in Lemma 4.2.6. □

We have already seen that the rank of $NS(\widetilde{V} \otimes \mathbb{C}, \mathbb{C})$ equals 20, so instead of the 27 generators given in lemma 4.2.13, we want 20 generators. First we extend the 27 generators to 33 by adding the last 3 exceptional curves E_1, E_2 and E_3 and the conics $\widetilde{D}_{B\pm\pm}$. This is done because these are defined over $\mathbb{Q}$. We filter these 33 generators to 20 by using intersection theory. The intersection numbers are relatively easy to compute. Clearly, we have $E_i \cdot E_j = 0$ if $i \neq j$. As $K_{\widetilde{V}} = 0$ is a canonical divisor and for the genus we have $g(E_j) = 0$, we find from the adjunction formula that $E_j^2 = -2$. Similarly, we have $\widetilde{D}^2 = -2$ for any of the 17 considered conics $\widetilde{D}$ on $\widetilde{V}$. Since all these conics are nonsingular, we find that

$$\widetilde{D} \cdot E_j = \begin{cases} 0 & \text{if } Q_j \text{ does not lie on } D = \pi(\widetilde{D}), \\ 1 & \text{if } Q_j \text{ lies on } D. \end{cases}$$

It remains to compute $\widetilde{D}_1 \cdot \widetilde{D}_2$ for 2 different conics out of the considered 17. It turns out that each pair D_1, D_2 out of the 17 considered conics on $V \otimes \mathbb{C}$ intersect each other either not at all, or transversally in 1 or 2 points. The points of intersection are in most cases singular points of $V \otimes \mathbb{C}$, so after blowing up at the singular points these conics do not intersect each other anymore on $\widetilde{V} \otimes \mathbb{C}$. The only pairs of conics that intersect each other in a nonsingular point of $V \otimes \mathbb{C}$ are

$$\begin{array}{cccc} D_{Aj_1j_2} & \text{and} & D_{Bl_1l_2}, & \text{with } j_1j_2 = l_1l_2, \\ D_{U-+} & \text{and} & D_{Cj_1j_2}, & \text{with } j_1j_2 = 1. \end{array}$$

Hence the 10 pairs

$$\begin{array}{cccc} \widetilde{D}_{Aj_1j_2} & \text{and} & \widetilde{D}_{Bl_1l_2}, & \text{with } j_1j_2 = l_1l_2, \\ \widetilde{D}_{U-+} & \text{and} & \widetilde{D}_{Cj_1j_2}, & \text{with } j_1j_2 = 1 \end{array}$$

still intersect on $\widetilde{V} \otimes \mathbb{C}$ and their intersection number is 1. For all other pairs out of the 17 considered conics on $\widetilde{V} \otimes \mathbb{C}$ the intersection number equals 0. We now have all $33^2 = 1089$ intersection numbers which give the intersection matrix. Indeed, it has rank 20 as expected. From the intersection matrix we also find that 13 of the 33 given classes in $NS(\widetilde{V} \otimes \mathbb{C}, \mathbb{C})$ are $\mathbb{Z}$-linear combinations of the 20 others. Hence the latter 20 form a set of generators for $NS(\widetilde{V} \otimes \mathbb{C}, \mathbb{C})$. We find the following proposition.

Proposition 4.2.14 *The Néron-Severi group $NS(\widetilde{V} \otimes \mathbb{C}, \mathbb{C})$ of $\widetilde{V} \otimes \mathbb{C}$ has rank 20 and is generated by the classes of the exceptional curves E_1, E_3, E_4, E_5, E_6, E_7, E_8, E_{10}, E_{12}, E_{13} and the conics $\widetilde{D}_{A\pm-}$, $\widetilde{D}_{Y++}$, $\widetilde{D}_{U-+}$, $\widetilde{D}_{C++}$, $\widetilde{D}_{C+-}$, $\widetilde{D}_{C-+}$, $\widetilde{D}_{B++}$, $\widetilde{D}_{B+-}$, $\widetilde{D}_{B-+}$. Apart from the 5 divisors E_{10}, E_{12}, E_{13}, $\widetilde{D}_{Y++}$ and $\widetilde{D}_{U-+}$ these are all defined over $\mathbb{Q}$.*

Remark 4.2.15 The intersection matrix of these 20 divisors has determinant -16 and is given by

	$\widetilde{D}_{A+-}$	$\widetilde{D}_{A--}$	$\widetilde{D}_{B++}$	$\widetilde{D}_{B+-}$	$\widetilde{D}_{B-+}$	$\widetilde{D}_{C++}$	$\widetilde{D}_{C+-}$	$\widetilde{D}_{C-+}$	$\widetilde{D}_{Y++}$	$\widetilde{D}_{U-+}$	E_1	E_3	E_4	E_5	E_6	E_7	E_8	E_{10}	E_{12}	E_{13}
$\widetilde{D}_{A+-}$	-2	0	0	1	1	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0
$\widetilde{D}_{A--}$	0	-2	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0
$\widetilde{D}_{B++}$	0	1	-2	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1
$\widetilde{D}_{B+-}$	1	0	0	-2	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1
$\widetilde{D}_{B-+}$	1	0	0	0	-2	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0
$\widetilde{D}_{C++}$	0	0	0	0	0	-2	0	0	0	1	1	0	0	1	1	0	0	0	0	0
$\widetilde{D}_{C+-}$	0	0	0	0	0	0	-2	0	0	0	0	1	1	1	1	0	0	0	0	0
$\widetilde{D}_{C-+}$	0	0	0	0	0	0	0	-2	0	0	1	0	0	0	0	1	1	0	0	0
$\widetilde{D}_{Y++}$	0	0	0	0	0	0	0	0	-2	0	1	1	0	0	0	0	0	0	0	0
$\widetilde{D}_{U-+}$	0	0	0	0	0	1	0	0	0	-2	0	0	0	0	0	0	0	1	1	0
E_1	0	0	0	0	0	1	0	1	1	0	-2	0	0	0	0	0	0	0	0	0
E_3	1	1	0	0	0	0	1	0	1	0	0	-2	0	0	0	0	0	0	0	0
E_4	0	0	0	0	0	0	1	0	0	0	0	0	-2	0	0	0	0	0	0	0
E_5	0	0	1	0	1	1	1	0	0	0	0	0	0	-2	0	0	0	0	0	0
E_6	0	0	0	1	0	1	1	0	0	0	0	0	0	0	-2	0	0	0	0	0
E_7	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	-2	0	0	0	0
E_8	0	0	1	0	1	0	0	1	0	0	0	0	0	0	0	0	-2	0	0	0
E_{10}	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	-2	0	0
E_{12}	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	-2	0
E_{13}	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2

5 What more to do on perfect cuboids

So far we still haven't found a perfect cuboid, nor a proof of nonexistence. In section 3.4 we have seen that conjecturally there are few rational points on the surface Υ, where "few" means that the set of rational points on Υ is not Zariski dense.

Although it is highly forbidden to use probability theory on numbers, if we were to use it anyway, we would find that the expected number of perfect cuboids with all sides greater than a fixed number x is less than $x^{-3/2}$. Together with the fact that from computer searches we know that there are no perfect cuboids with any side less than 10^9, this gives at least psychological support to believe that there are indeed "few" perfect cuboids.

Obviously, there are basically two roads to take to get any further with the question of the (non)existence of perfect cuboids. One could try to find either a concrete perfect cuboid or a proof of nonexistence. We will now state some ideas that might help to find an approach to the problem.

Idea 1. We can use Proposition 4.2.14 to find out more about the rational curves on $\widetilde{V}$. For any rational curve C found on $\widetilde{V}$ we can try to find rational points on C that lift to a rational point on Υ. First of all we could try to find all rational curves on $\widetilde{V}$ of small degree, just as A. Bremner does in [38].

We could also try to follow a result of H. Sterk [58], stating the following. Let $\mathrm{Aut}(\widetilde{V}\otimes\mathbb{C})$ be the group of (biholomorphic) automorphisms of the K3 surface $\widetilde{V}\otimes\mathbb{C}$. Then $\mathrm{Aut}(\widetilde{V}\otimes\mathbb{C})$ is finitely generated and the number of $\mathrm{Aut}(\widetilde{V}\otimes\mathbb{C})$-orbits in the collection of complete linear systems which contain an irreducible rational curve is finite. It may be possible to find all these orbits and their irreducible rational curves.

Idea 2. As long as we don't know whether the surface Υ contains any non-trivial rational points, we'd better have as much birationally equivalent surfaces at our disposal as possible. This might help both in case one tries to find perfect cuboids and in case one tries to prove nonexistence.

Apart from Υ and $\Upsilon/\langle\iota_Z\rangle$ that we have seen in the previous sections, one may want to study Υ/H for any subgroup H of $G\subset\mathrm{Aut}(\Upsilon)$. In [2] A. Bremner analysed a surface which was birationally equivalent with the surface $\Upsilon/\langle\iota_U\rangle$, see section 2. Some other interesting surfaces are the following.

W. Colman [5] shows that a perfect cuboid corresponds with a rational solution to the system of elliptic curves

$$w^4+2Aw^3+2w^2-2Aw+1=s^2,\qquad w^4+\frac{8}{A}w^3+2w^2-\frac{8}{A}w+1=t^2,\tag{43}$$

fibered over the conic $AD=D^2+1$.

It is sometimes convenient to work with a hypersurface instead of a variety of higher codimension. The rational map

$$\begin{cases} A=uw(v^2-z^2),\\ B=vw(u^2-z^2),\\ C=2uvzw,\\ X=vw(u^2+z^2),\\ Y=uw(v^2+z^2),\\ Z=uv(z^2-w^2),\\ U=uv(z^2+w^2)\end{cases}\qquad\text{or}\qquad\begin{cases} u=C(Y-A)(U+Z),\\ v=C(X-B)(U+Z),\\ w=C(X-B)(Y-A),\\ z=(X-B)(Y-A)(U+Z),\end{cases}$$

shows that Υ is birationally equivalent with the hypersurface in $\mathbb{P}^3$ given by

$$(z^2+w^2)^2u^2v^2=w^2(u^2+v^2)(z^4+u^2v^2).$$

Lots of automorphisms are revealed by rewriting this equation as

$$\left(\frac{u}{z}-\frac{z}{u}\right)^2+\left(\frac{v}{z}-\frac{z}{v}\right)^2=\left(\frac{w}{z}-\frac{z}{w}\right)^2, \tag{44}$$

Idea 3. We have seen in section 2 that Leech [18] states that there are no rational cuboids with sides A, B and C such that $A:B=4:3$. This is done by considering equation (5). Note that we can assume that $A:B=2ab:(a^2-b^2)$. Substituting $u=\alpha/\beta$ and $v=a_2/b_2$ this equation becomes

$$\frac{a^2-b^2}{2ab}u(v^2-1)=v(u^2-1),$$

an elliptic curve with parameter a/b. Using an infinite descent it can be shown that for $a/b=2$ this elliptic curve has no nontrivial rational points. It follows that there are no rational cuboids with sides A, B, C under the extra constraint $A:B=4:3$.

Using the same method we can prove nonexistence of rational cuboids under similar constraints. It may also be interesting to prove nonexistence of rational or perfect cuboids under various other types of extra constraints.

Idea 4. If we want to search for perfect cuboids, then that would of course be done by computer. Apart from inventing new search algorithms, there are several algorithms, described in section 2, that might be made much faster by considering p-adic constraints. Such constraints may be found by imitating N. Elkies' search for solutions of the equation $a^4+b^4+c^4=d^4$ in [44]. Note that we will not be able to prove that Υ has no nontrivial points over the p-adic numbers $\mathbb{Z}_p$ for some prime p. Indeed, for $x\in\mathbb{Z}_p$ with $x\equiv 1(\bmod p^n)$ for n large enough ($n\geq 1$ for $p>2$ and $n\geq 3$ for $p=2$), we know that x is a square in $\mathbb{Z}_p$. Hence the cuboid with sides 1, p^{2n} and p^{4n} is "perfect" over $\mathbb{Z}_p$.

6 References

6.1 References about cuboids

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