Geometry dictates arithmetic

Ronald van Luijk

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Utrecht
Curves

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Definition.

Genus of a smooth projective curve $C$ over $\mathbb{Q}$ is the genus of $C(\mathbb{C})$.

<table>
<thead>
<tr>
<th>$g = 0$</th>
<th>$g = 1$</th>
<th>$g \geq 2$</th>
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<tbody>
<tr>
<td><img src="image1.png" alt="Image" /></td>
<td><img src="image2.png" alt="Image" /></td>
<td><img src="image3.png" alt="Image" /></td>
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Genus 0

$\mathbb{P}^2 \ni C : x^2 + y^2 = 1$

Points on the curve:
- $(0, 1)$
- $(\frac{4}{5}, \frac{3}{5})$
- $(1, 0)$
Genus 0

\[ \mathbb{P}^2 \ni C : x^2 + y^2 = 1 \]
Theorem. If a conic over $\mathbb{Q}$ has a rational point, then it has infinitely many.

Theorem. If a conic $D$ over $\mathbb{Q}$ has a rational point, then there is an isomorphism $\mathbb{P}^1(\mathbb{C}) \to D(\mathbb{C})$, so the genus of $D$ is 0.

Theorem. Any curve of genus 0 over $\mathbb{Q}$ is isomorphic to a conic.

Theorem. If a curve of genus 0 over $\mathbb{Q}$ has a rational point, then it is isomorphic to $\mathbb{P}^1$ and it has infinitely many rational points.

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$\begin{align*}
\mathbb{P}^2 & \supset C : x^2 + y^2 = 1 \\
& \text{Points: } (0, 1), (1, 0), (-1, 0), \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)
\end{align*}$
Theorem. If a conic over \( \mathbb{Q} \) has a rational point, then it has infinitely many.

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Theorem. If a conic $D$ over $\mathbb{Q}$ has a rational point, then there is an isomorphism $P_1(\mathbb{C}) \rightarrow D(\mathbb{C})$, so the genus of $D$ is 0.

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$$N_C(B) = \text{number of } \left( \frac{a}{c}, \frac{b}{c} \right) \text{ with } |a|, |b|, |c| \leq B$$
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\[ N_C(B) \sim \frac{4}{\pi} \cdot B \]
Theorem. The number $N_D(B)$ of rational points on a conic $D$ grows linearly with the height $B$ (or is zero).

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Fact. \( E(k) \) is an abelian group!

Theorem (Mordell-Weil). For any elliptic curve \( E \) over \( \mathbb{Q} \), the group \( E(\mathbb{Q}) \) is finitely generated.

Here: \( \text{rank} = 1 \), and \( \mathbb{Z} \cong E(\mathbb{Q}) = \langle (3, 1) \rangle \).

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\[N_E(B) = \text{number of } \left( \frac{a}{c}, \frac{b}{c} \right) \text{ with } |a|, |b|, |c| \leq B\]

\[N_E(B) \sim \gamma \sqrt{\log B}\]

\[\gamma = 2.6768125\ldots\]
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Number of \((a, b, c)\) with \(|a|, |b|, |c| \leq B\)

Theorem. For any elliptic curve \(E\) over \(\mathbb{Q}\) with \(r = \text{rank } E(\mathbb{Q})\), we have

\[N_E(B) \sim c (\log B)^{r/2}.\]
Genus $g \geq 2$

Examples.

- $y^2 = f(x)$ with $f$ separable of degree $2g + 2$.
- Smooth projective plane curve of degree $d \geq 4$ with $g = \frac{1}{2}(d - 1)(d - 2)$. 

Theorem ("Mordell Conjecture" by Faltings, 1983).

Any curve over $\mathbb{Q}$ with $g \geq 2$ has only finitely many rational points.

Conclusion.

"The higher the genus, the lower the number of rational points."
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Definition.
Let \( X \) be a smooth projective variety with function field \( k(X) \). Then \( \Omega_{k(X)/k} \) is the \( k(X) \)-vectorspace of differential 1-forms, generated by \( \{ df : f \in k(X) \} \) and satisfying

\[
\begin{align*}
\text{d}(f + g) &= df + dg, \\
\text{d}(fg) &= f dg + g df, \\
da &= 0 \text{ for } a \in k.
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Differentials

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Let $X$ be a smooth projective variety with function field $k(X)$. Then $\Omega_{k(X)/k}$ is the $k(X)$-vectorspace of differential 1-forms, generated by $\{df : f \in k(X)\}$ and satisfying

- $d(f + g) = df + dg$,
- $d(fg) = fdg + gdf$,
- $da = 0$ for $a \in k$.

Proposition. We have $\dim_{k(X)} \Omega_{k(X)/k} = \dim X$.

Example.
For curve $C: y^2 = f(x)$ we have $2ydy = f'(x)dx$ in $\Omega_{k(C)/k}$.
Holomorphic differentials on curves

Definition. For a point $P$ on a smooth projective curve $C$ with local parameter $t_P \in k(C)$ and a differential $\omega \in \Omega_{k(C)/k}$, we write $\omega = f_P dt_P$; then $\omega$ is holomorphic at $P$ if $f_P$ has no pole at $P$.

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Curve $C : y^2 = f(x)$ with $f$ separable of degree $d \geq 3$. Then

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**Definition.** Set $\Omega_{C/k} = \{ \omega \in \Omega_{k(C)/k} : \omega \text{ holom. everywhere} \}$.

**Proposition.** We have $g = \dim_k \Omega_{C/k}$.
Holomorphic differentials in general

Recall. If $X$ smooth, projective, then $\dim_{k(X)} \Omega_{k(X)/k} = \dim X$.

Fact. If $V$ is a vector space with $\dim V = n$, then $\dim \bigwedge^n V = 1$. 
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Definition (unconventional notation for $(\dim X)$-forms).
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For a $k$-basis $(\omega_0, \omega_1, \ldots, \omega_N)$ of $\Omega_{X/k}$, we get $f_i \in k(X)$ such that $\omega_i = f_i \omega_0$. The Kodaira dimension $\kappa(X)$ of $X$ is $-1$ if $\dim_k \Omega_{X/k} = 0$, or the dimension of the image of the map $X \to \mathbb{A}^N$, $P \mapsto (f_1(P), f_2(P), \ldots, f_N(P))$. 
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$$X \to \mathbb{A}^N, \quad P \mapsto (f_1(P), f_2(P), \ldots, f_N(P)).$$

Proposition. For a curve $C$ we get

$$\kappa(C) = \begin{cases} 
-1 & g = 0 \\
0 & g = 1 \\
1 & g \geq 2
\end{cases}$$
Varieties of general type

In general, $-1 \leq \kappa(X) \leq \dim X$ (complex $X \Rightarrow$ high $\kappa(X)$).

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**Conjecture** (Lang).
If $X$ is a variety over $\mathbb{Q}$ that is of general type, then the rational points lie in a Zariski closed subset, i.e., a finite union of proper subvarieties of $X$. 
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**Corollary.** Let $X \subset \mathbb{P}^3$ be a smooth, projective surface over $\mathbb{Q}$ of degree $\geq 5$. Then the rational points are all contained in some finite union of curves.
Fano varieties

**Definition.** A Fano variety is a smooth, projective variety $X$ with ample anti-canonical bundle.

We have $\kappa(X) = -1$ and $X$ is geometrically “easy”.

Conjecture (Batyrev-Manin). Suppose $X$ over $\mathbb{Q}$ is Fano. Set $\rho = \text{rk} \text{Pic} X$.

There is an open subset $U \subset X$ and a constant $c$ with $N_{U}(B) \sim cB(\log B)^{\rho - 1}$.

This is proved in many cases for surfaces. False in higher dimension, but no counterexamples to lower bound.

**Conclusion.** The more complex a variety, the fewer rational points.
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**Examples**

- Smooth quartic surfaces in $\mathbb{P}^3$.
- Double cover of $\mathbb{P}^2$ ramified over a smooth sextic.
- Desingularization of $A/\langle[-1]\rangle$ for an abelian surface $A$. 
Theorem (Tschinkel-Bogomolov).
If \( \text{rk Pic } X \geq 5 \), then there is a finite extension \( K \) of \( \mathbb{Q} \) such that the \( K \)-rational points are Zariski dense on \( X \), i.e., rational points are potentially dense on \( X \).
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Question. Is there a K3 surface \( X \) over a number field \( K \) with \( X(K) \) neither empty nor dense?
K3 surfaces

**Theorem** (Logan, McKinnon, vL).
Take $a, b, c, d \in \mathbb{Q}^*$ with $abcd \in (\mathbb{Q}^*)^2$. Let $X \subset \mathbb{P}^3$ be given by

$$ax^4 + by^4 + cz^4 + dw^4.$$ 

If $P \in X(\mathbb{Q})$ has no zero coordinates and $P$ does not lie on one of the 48 lines (no two terms sum to 0), then $X(\mathbb{Q})$ is Zariski dense.
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**Question.** Are the conditions on \(P\) necessary?

**Conjecture** (vL) Every \(t \in \mathbb{Q}\) can be written as
\[
t = \frac{x^4 - y^4}{z^4 - w^4}.
\]
\[ S: x^3 - 3x^2y^2 + 4x^2yz - x^2z^2 + x^2z - xy^2z - xyz^2 + x \]
\[ + y^3 + y^2z^2 + z^3 = 0 \]

\[ N \sim 13.5 \cdot \log B \]
K3 surfaces

Conjecture (vL).
Suppose $X$ is a K3 surface over $\mathbb{Q}$ with $\text{rk} \, \text{Pic} \, X_\mathbb{C} = 1$. There is an open subset $U \subset X$ and a constant $c$ such that

$$N_U(B) \sim c \log B.$$
Theorem (Hasse). Let $Q \subset \mathbb{P}^n$ be a smooth quadric over $\mathbb{Q}$. Suppose that $Q$ has points over $\mathbb{R}$ and over $\mathbb{Q}_p$ for every $p$. Then $Q(\mathbb{Q}) \neq \emptyset$. 
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Proposition (Selmer).
The curve $C \subset \mathbb{P}^2$ given by $3x^3 + 4y^3 + 5z^3 = 0$ has points over $\mathbb{R}$ and over $\mathbb{Q}_p$ for every $p$, but $C(\mathbb{Q}) = \emptyset$. 
Brauer-Manin obstruction

To every variety $X$ we can assign the Brauer group $Br X$. Every morphism $X \to Y$ induces a homomorphism $Br Y \to Br X$. For every point $P$ over a field $k$ we have $Br(P) = Br(k)$. 

Corollary. If $(\prod_v X(Q_v))_{Br} := \phi^{-1}(0)$ is empty, then $X(Q) = \emptyset$.

Conjecture (Colliot-Thélène). This Brauer-Manin obstruction is the only obstruction to the existence of rational points for rationally connected varieties.
To every variety $X$ we can assign the Brauer group $\text{Br } X$. Every morphism $X \to Y$ induces a homomorphism $\text{Br } Y \to \text{Br } X$. For every point $P$ over a field $k$ we have $\text{Br}(P) = \text{Br}(k)$.

Let $X$ be smooth and projective.

Diagram:

\[
\begin{align*}
X(\mathbb{Q}) & \longrightarrow \prod_v X(\mathbb{Q}_v) \\
\downarrow & \quad \downarrow \phi \\
\text{Br}(\mathbb{Q}) & \longrightarrow \bigoplus_v \text{Br}(\mathbb{Q}_v) \longrightarrow \mathbb{Q}/\mathbb{Z}
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