An Elliptic K3 surface associated to Heron triangles

Ronald van Luijk
CRM, Montreal
MSRI, Berkeley

October 7, 2005
Queen's University

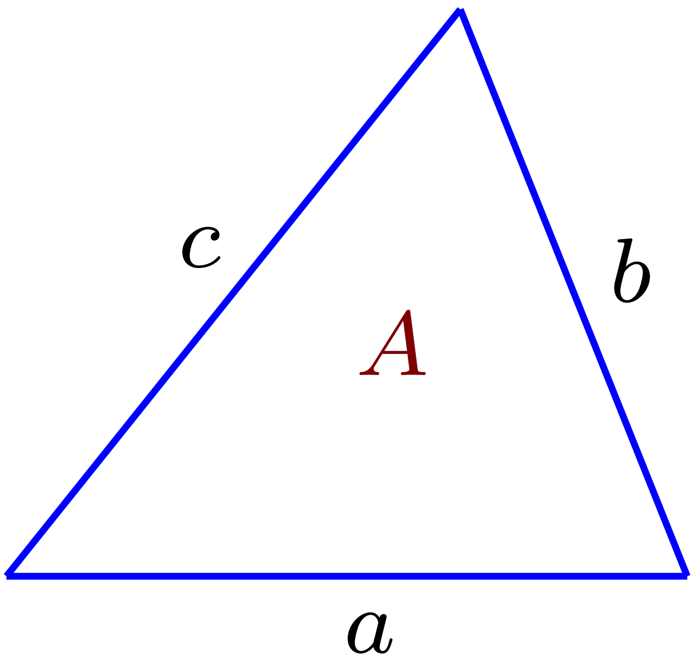
Goals:

(1) Solve a Diophantine problem

(2) Sketch a algebraic geometric point of view

(3) Open problems

Heron triangles

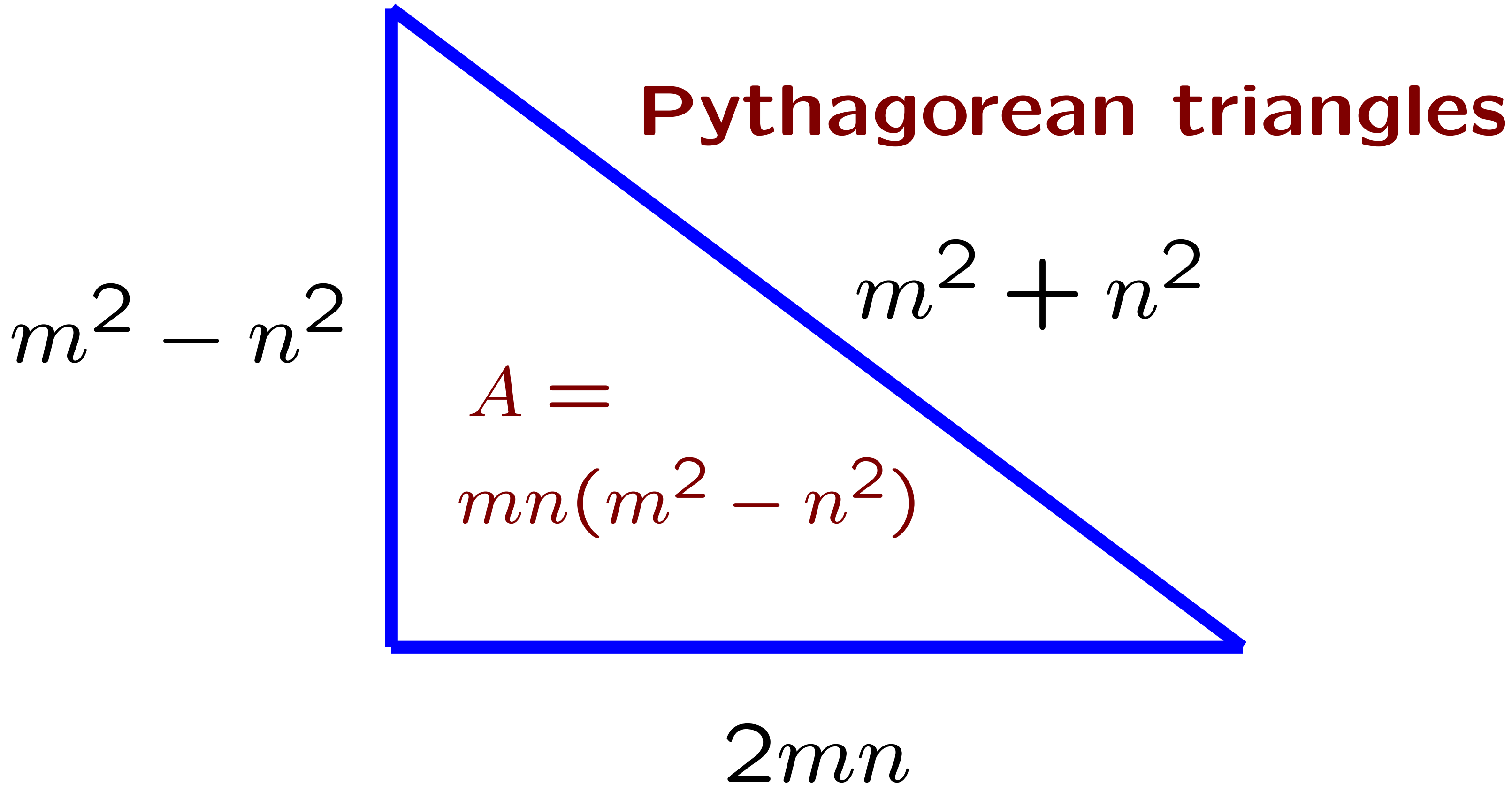


A **Heron triangle** is a triangle with integral sides and integral area.

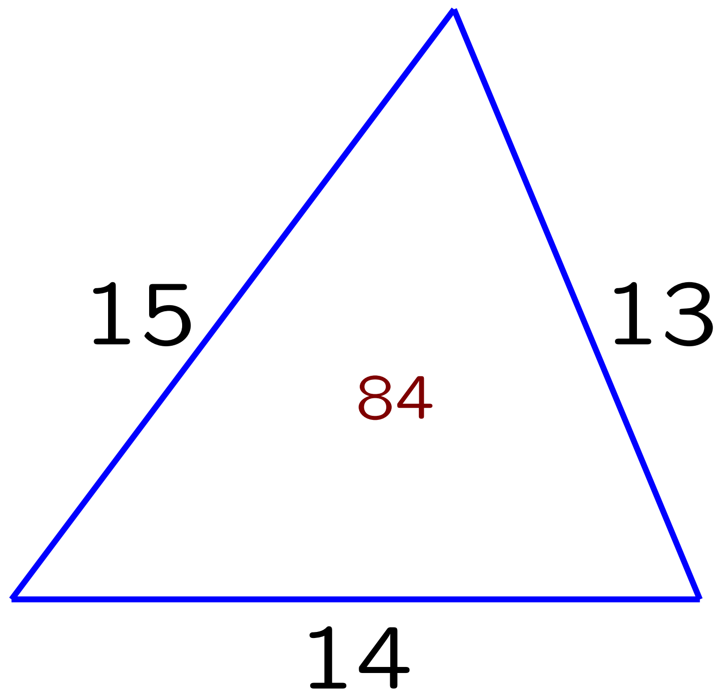
$$16A^2 = (a+b+c)(a+b-c)(a-b+c)(-a+b+c)$$

(Heron's formula)

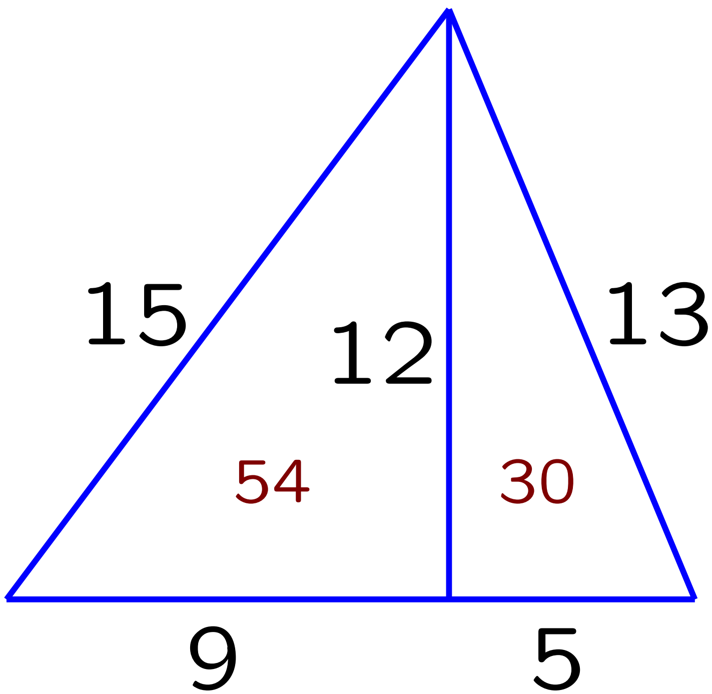
Examples of Heron triangles



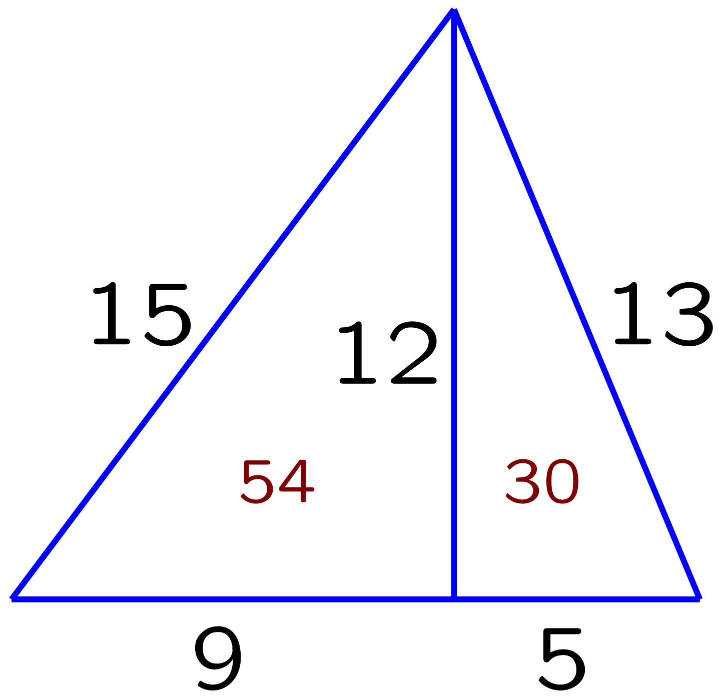
Examples of Heron triangles



Examples of Heron triangles



Examples of Heron triangles



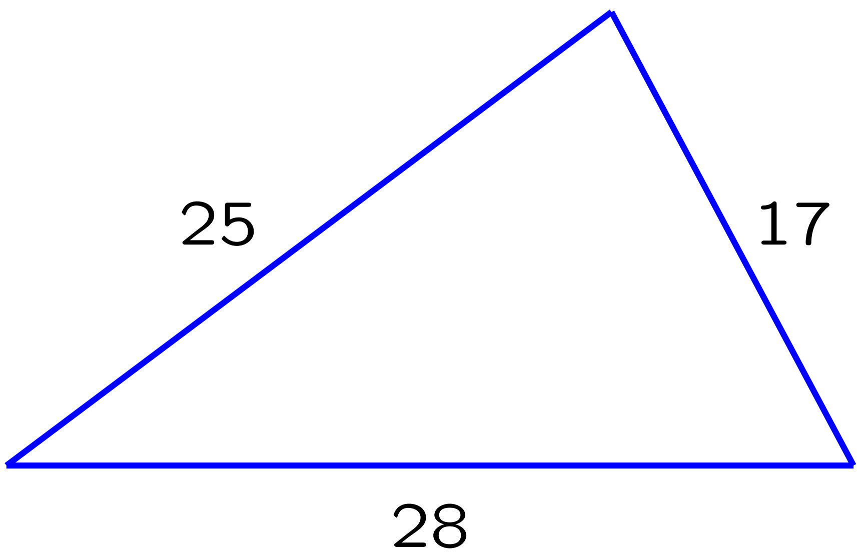
Fact All Heron triangles can be obtained by glueing together Pythagorean triangles.

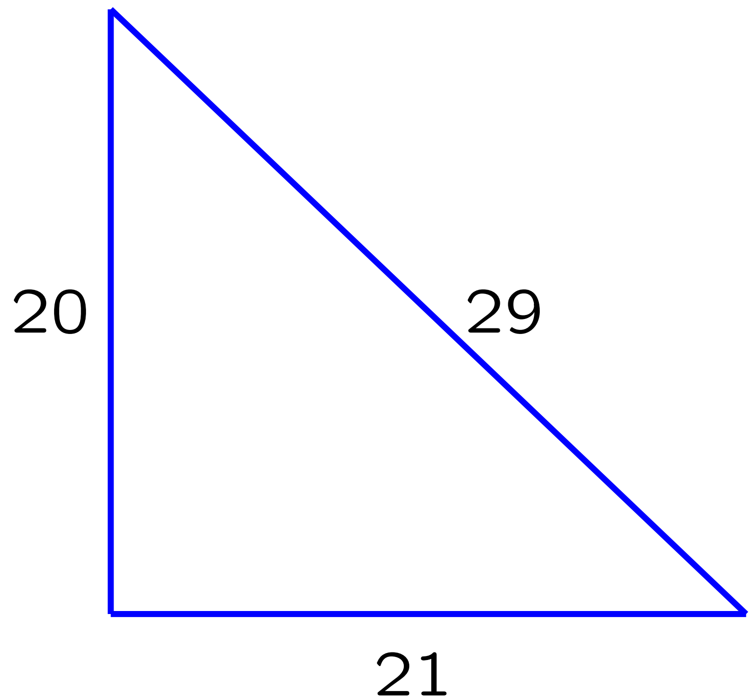
Examples of Heron triangles

$$\begin{aligned} a &= u(v^2 + w^2) & \quad 0 < v < u \\ b &= v(u^2 + w^2) & \quad 0 < w \\ c &= (u - v)(uv + w^2) \\ A &= uvw(u - v)(uv + w^2) \end{aligned}$$

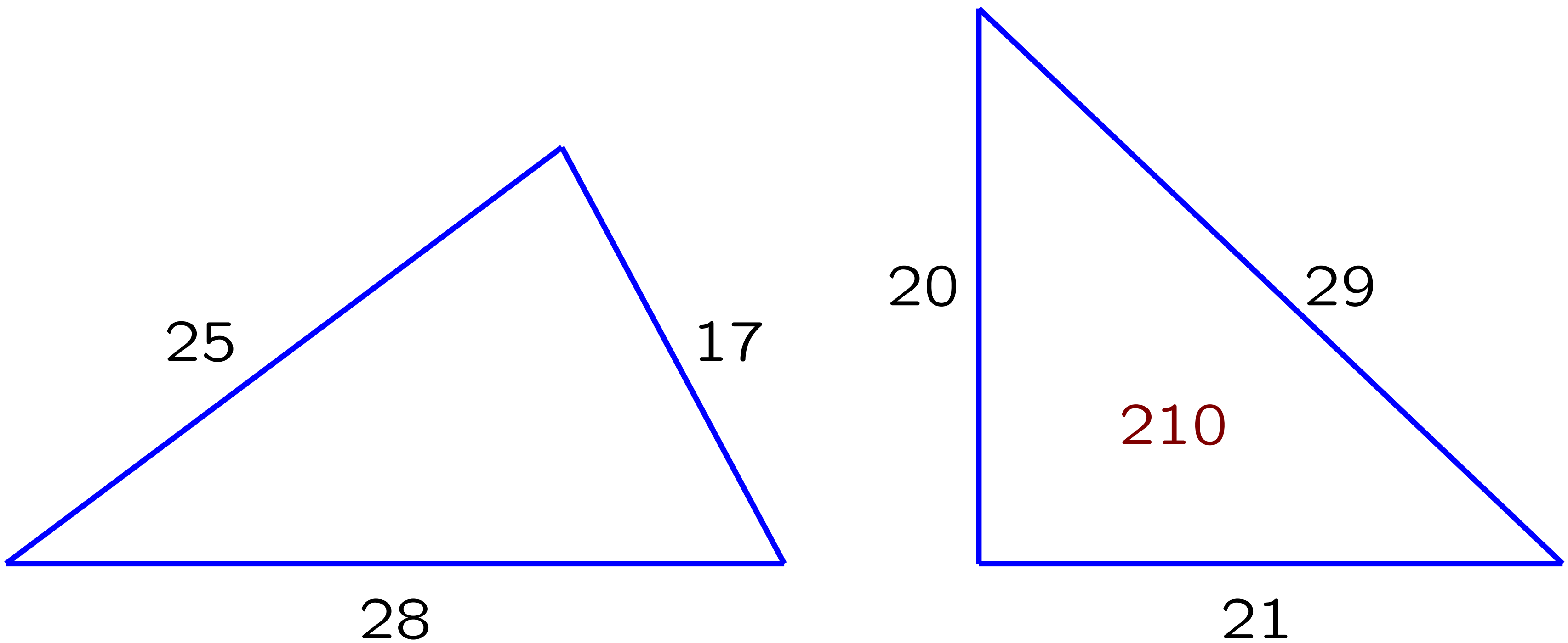
What more could we want than a parametrization?

Pairs of Heron triangles

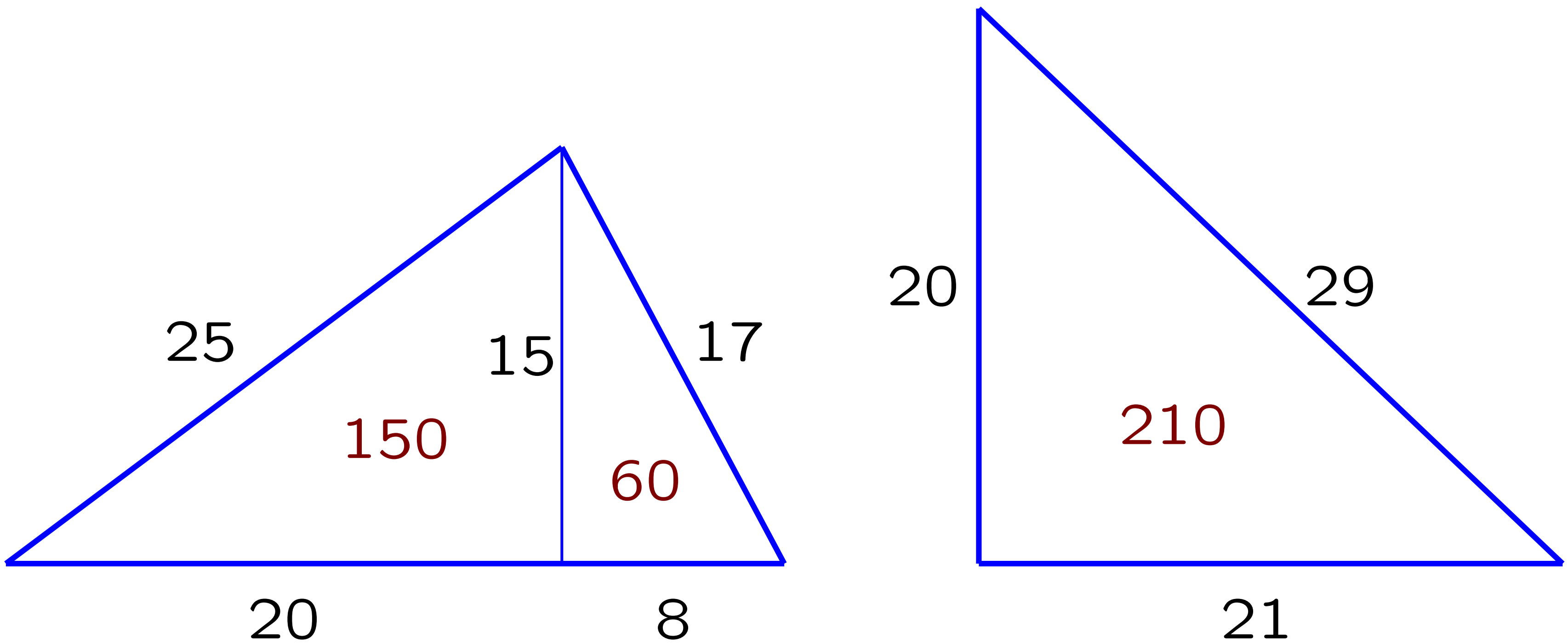




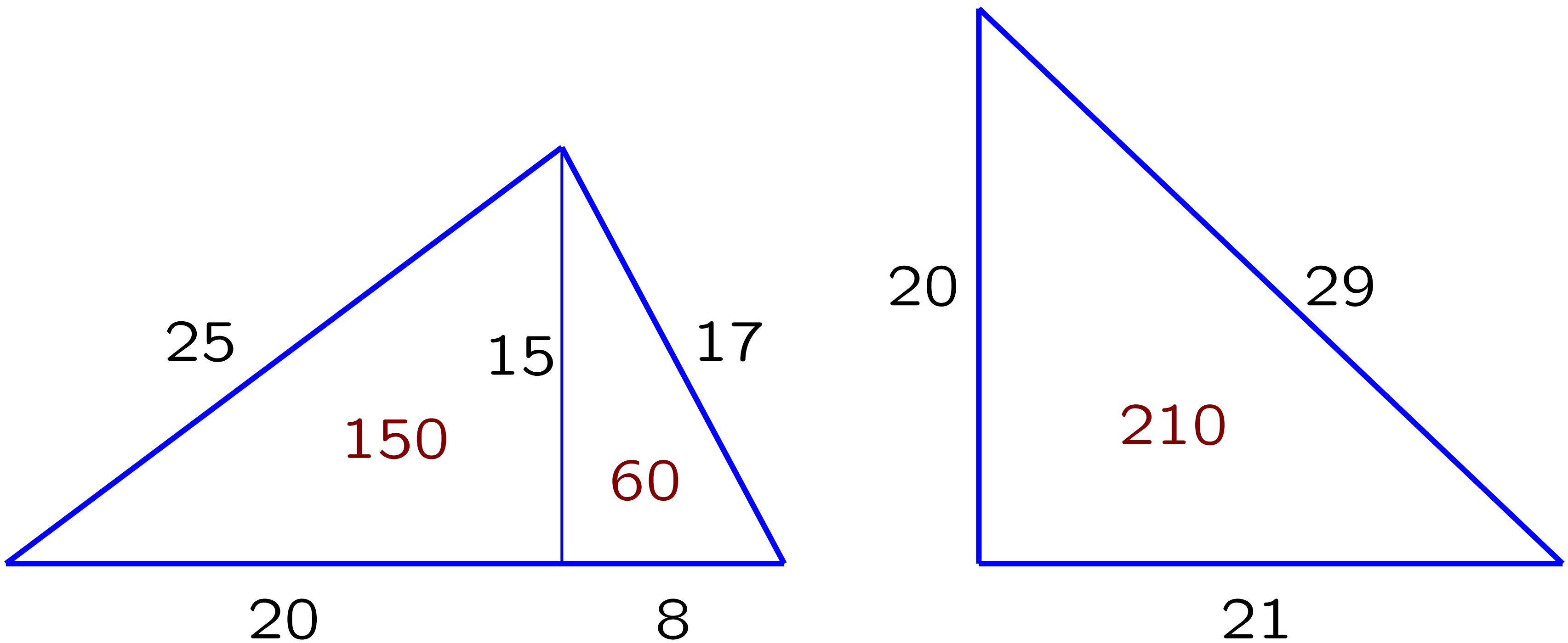
Pairs of Heron triangles



Pairs of Heron triangles



Pairs of Heron triangles



Same area and same perimeter!

Infinitely many pairs (Aassila, Kramer& Luca):

$$a_1 = t^{10} + 6t^8 + 15t^6 + 19t^4 + 11t^2 + 1$$
$$b_1 = t^{10} + 5t^8 + 10t^6 + 10t^4 + 6t^2 + 3$$
$$c_1 = t^8 + 5t^6 + 9t^4 + 7t^2 + 2$$

$$a_2 = t^{10} + 6t^8 + 15t^6 + 18t^4 + 9t^2 + 1$$
$$b_2 = t^{10} + 6t^8 + 14t^6 + 16t^4 + 9t^2 + 2$$
$$c_2 = t^6 + 4t^4 + 6t^2 + 3$$

$$p = 2t^{10} + 12t^8 + 30t^6 + 38t^4 + 24t^2 + 6$$
$$A = t(t^2 + 1)^4(t^2 + 2)(t^4 + 3t^3 + 3)$$

First Question:

Are there n-tuples of Heron triangles with the same area and the same perimeter for $n \geq 3$?

First Question:

Are there n-tuples of Heron triangles with the same area and the same perimeter for $n \geq 3$?

Answer:

Yes, in fact there are for any $n \geq 1$!

a	b	c
115439787835070058360 0		

Main Question:

Are there parametrizations of n-tuples of Heron triangles with the same area and the same perimeter for $n \geq 3$?

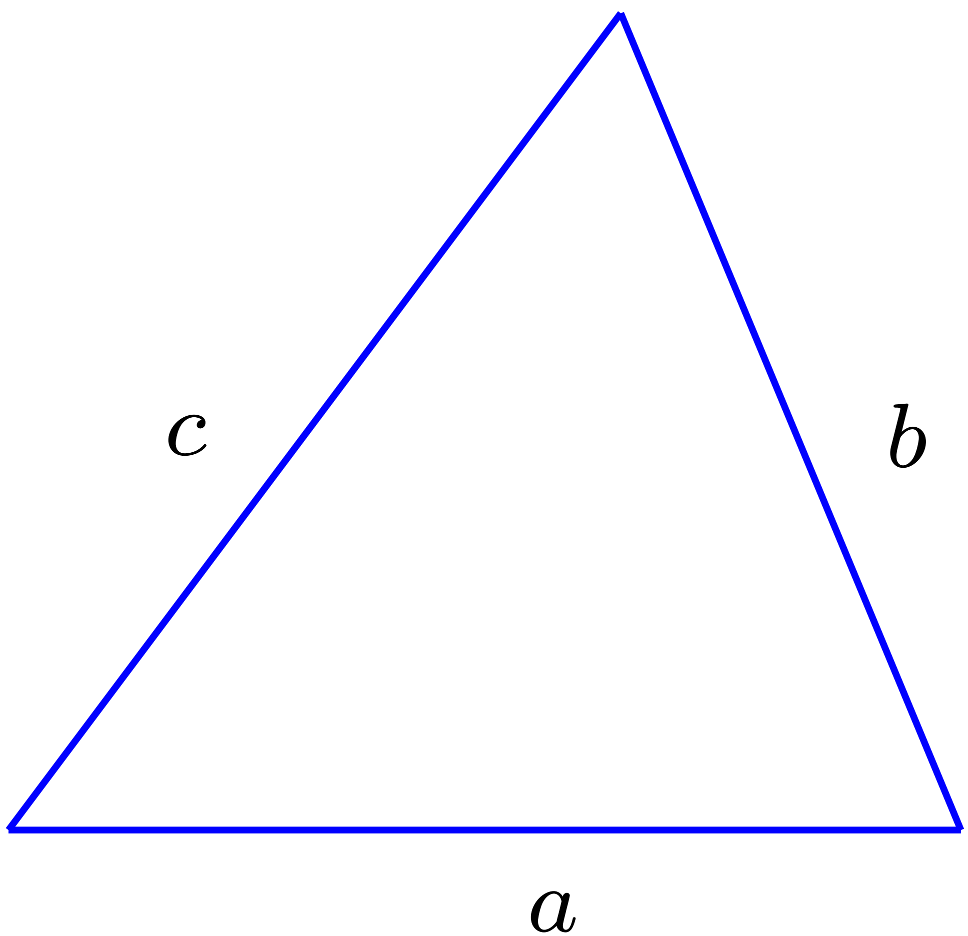
Main Question:

Are there parametrizations of n-tuples of Heron triangles with the same area and the same perimeter for $n \geq 3$?

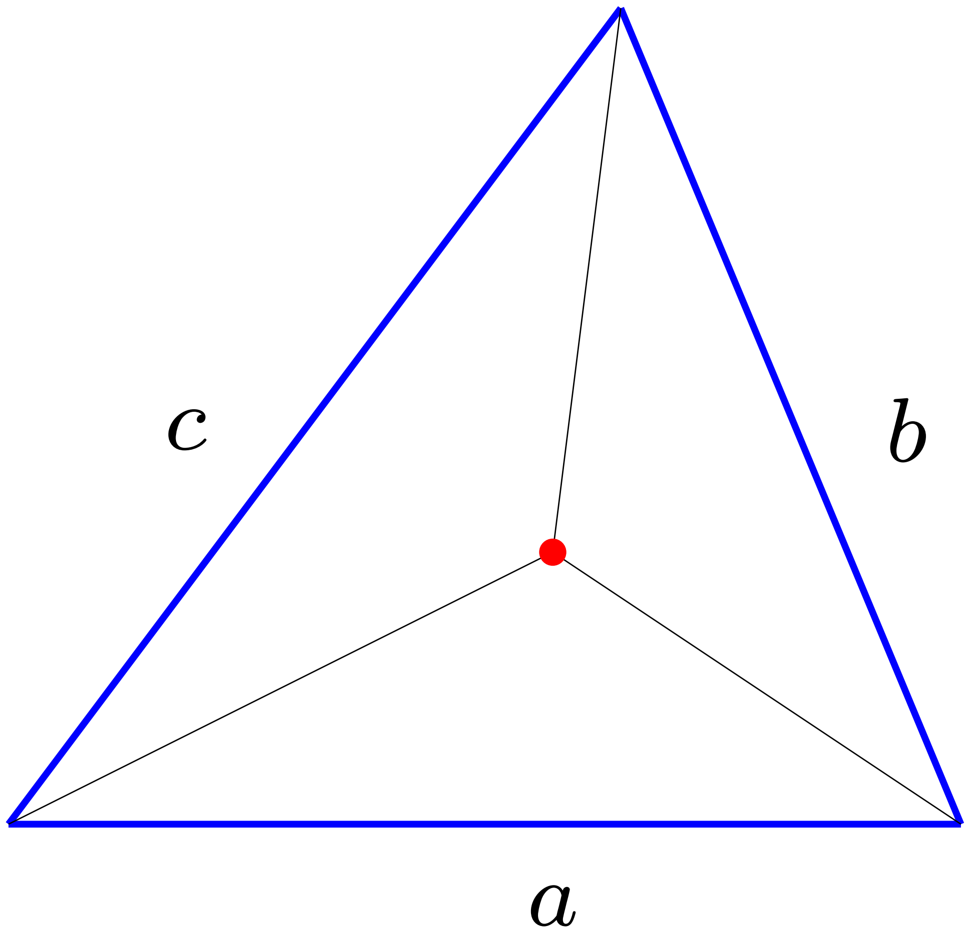
Answer:

Yes, such parametrizations exist for any n.

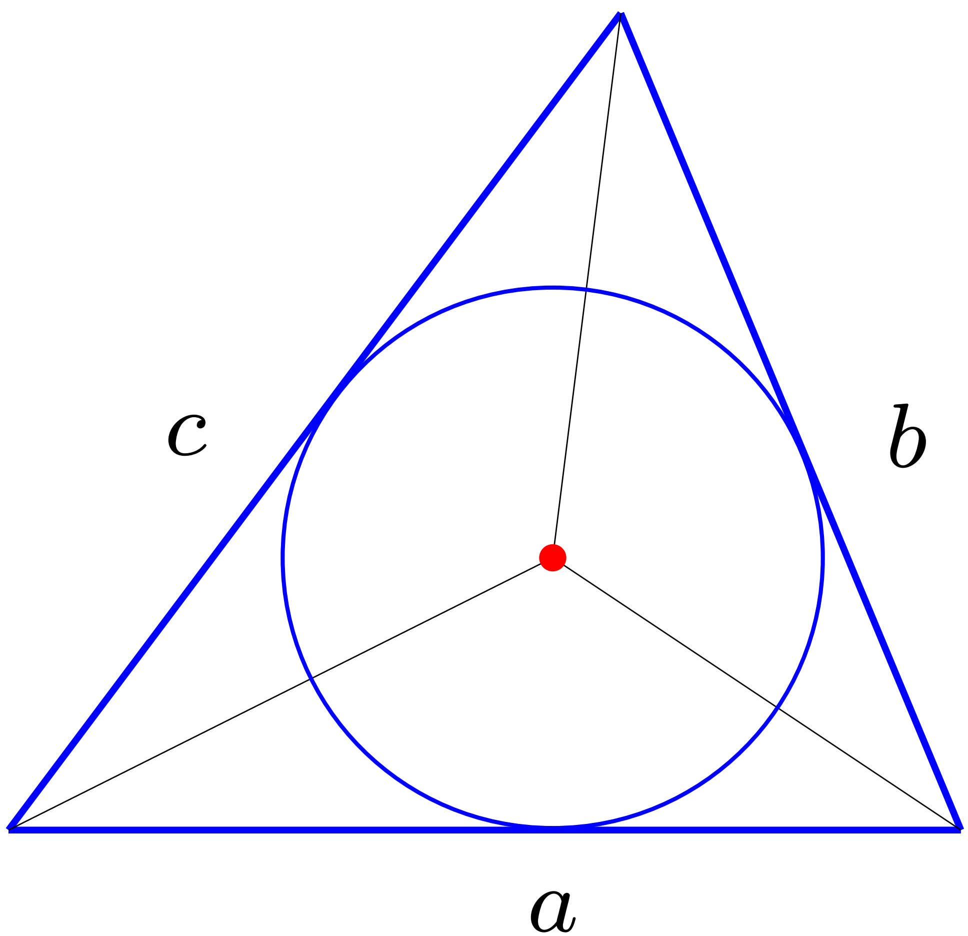
New variables



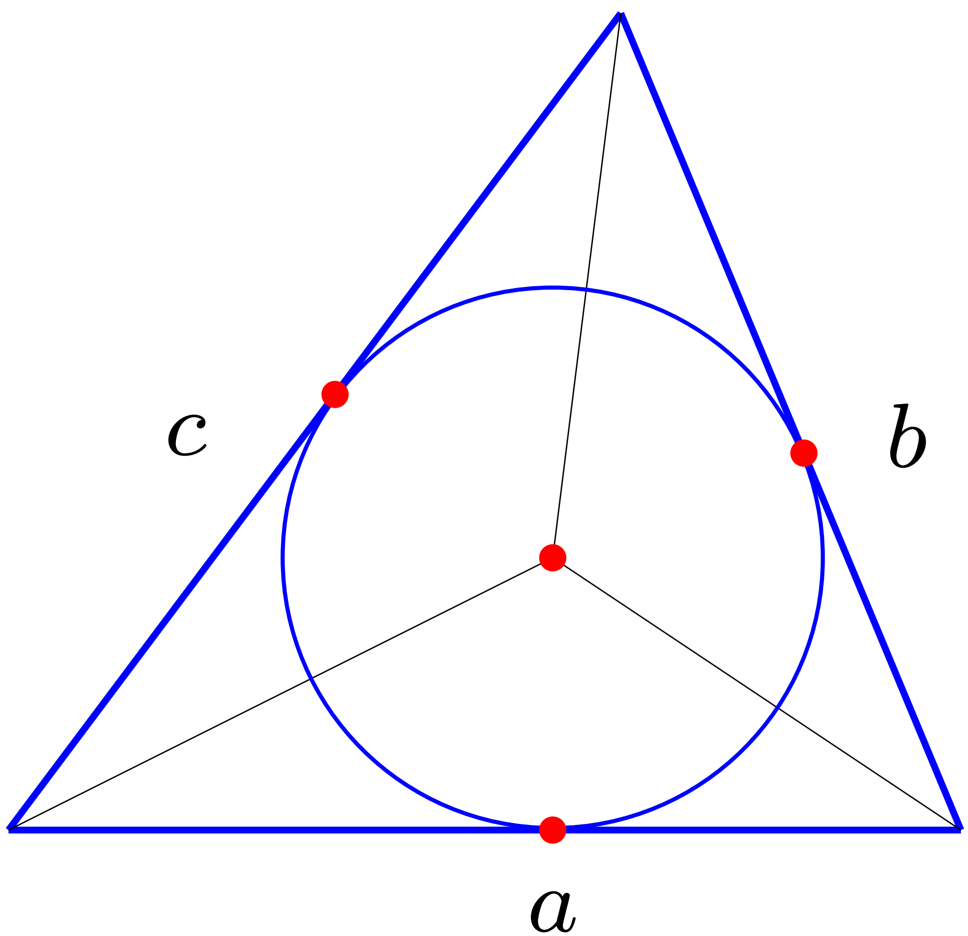
New variables



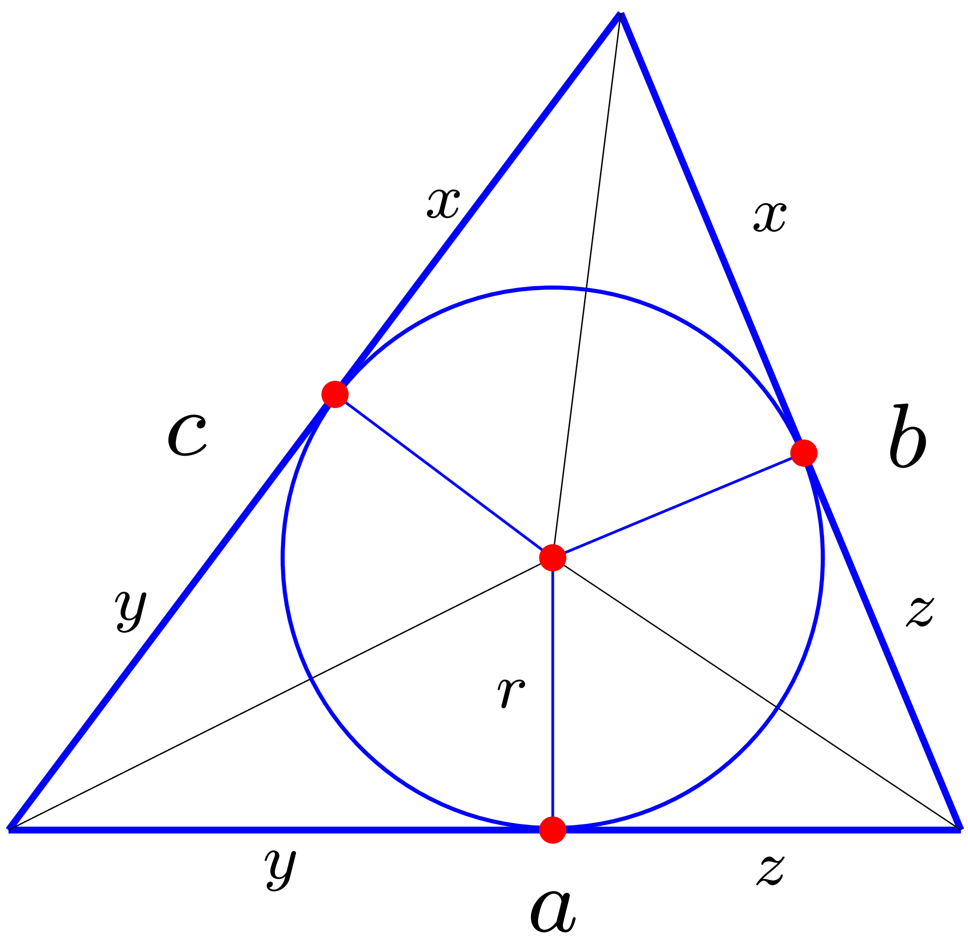
New variables



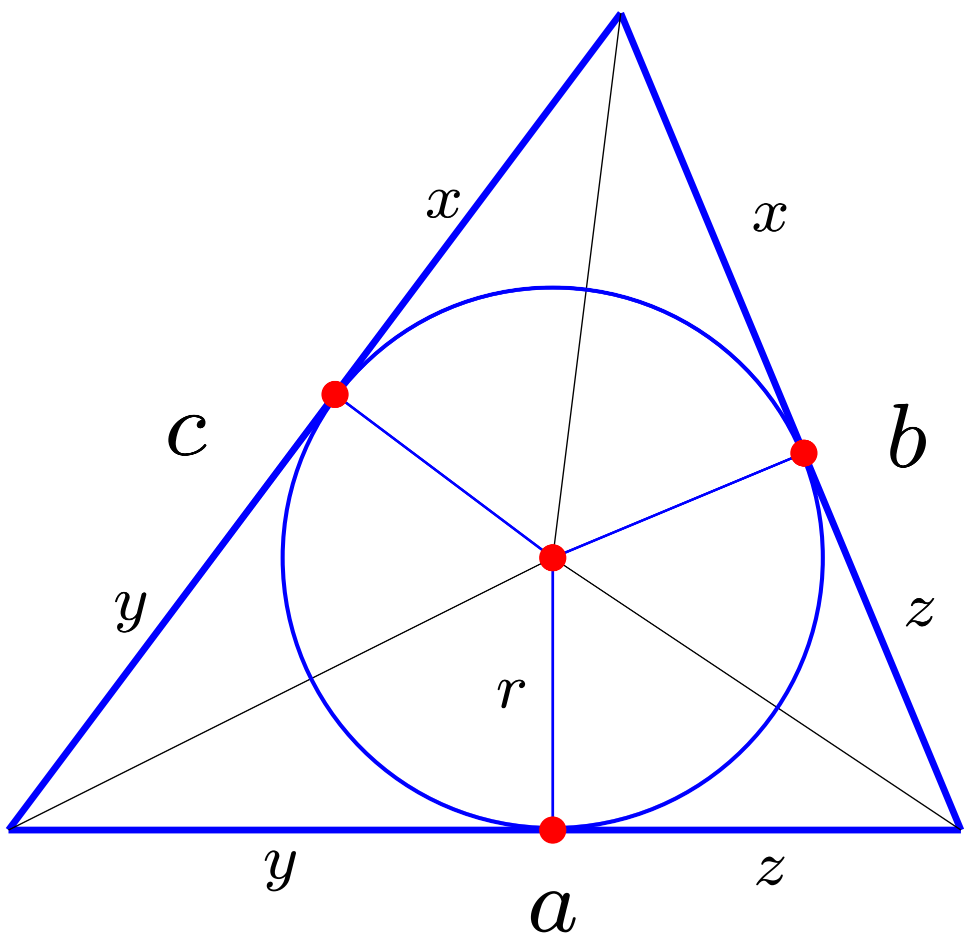
New variables



New variables

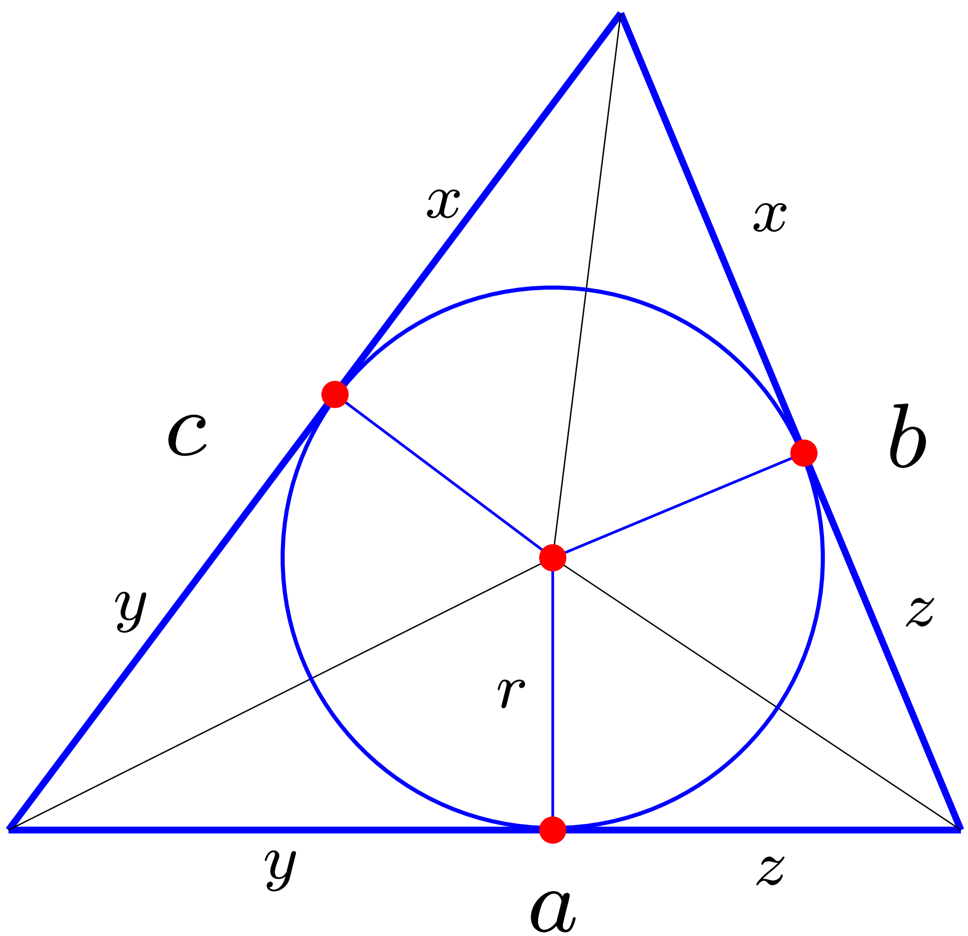


New variables



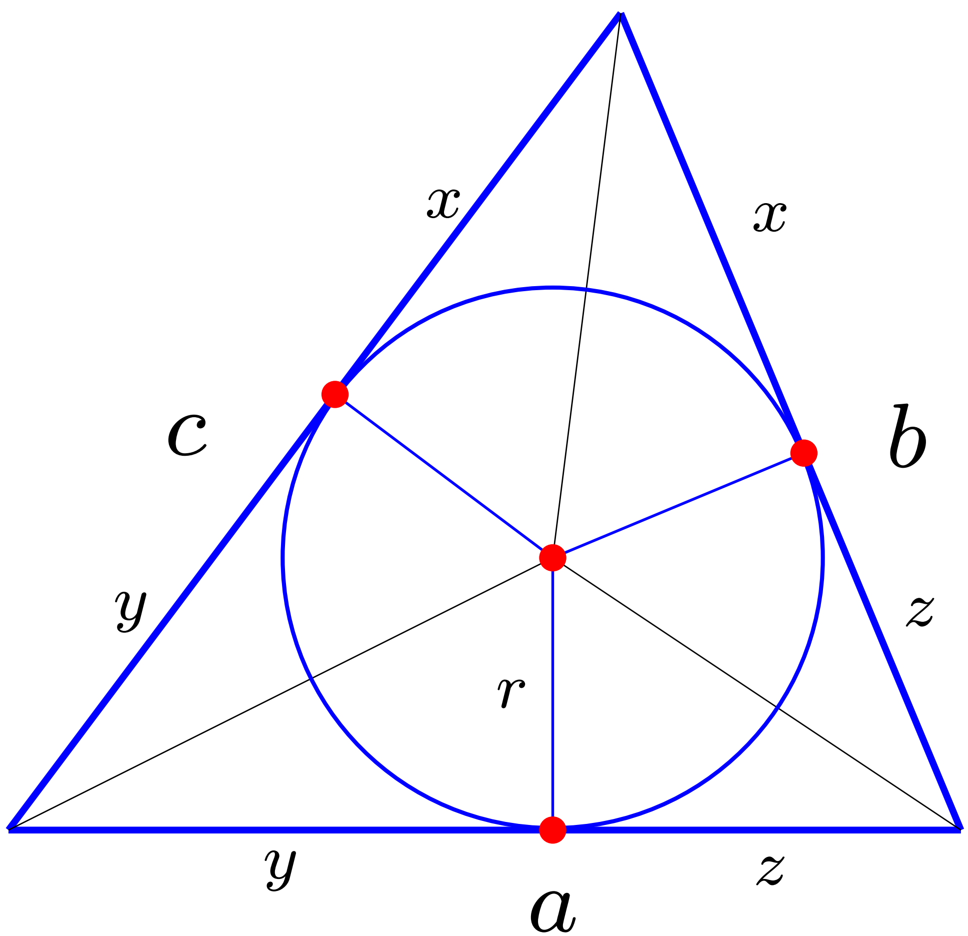
$$s = \frac{1}{2}(a + b + c)$$
$$s = x + y + z$$

New variables



$$s = \frac{1}{2}(a + b + c)$$
$$s = x + y + z$$
$$x = s - a$$
$$y = s - b$$
$$z = s - c$$

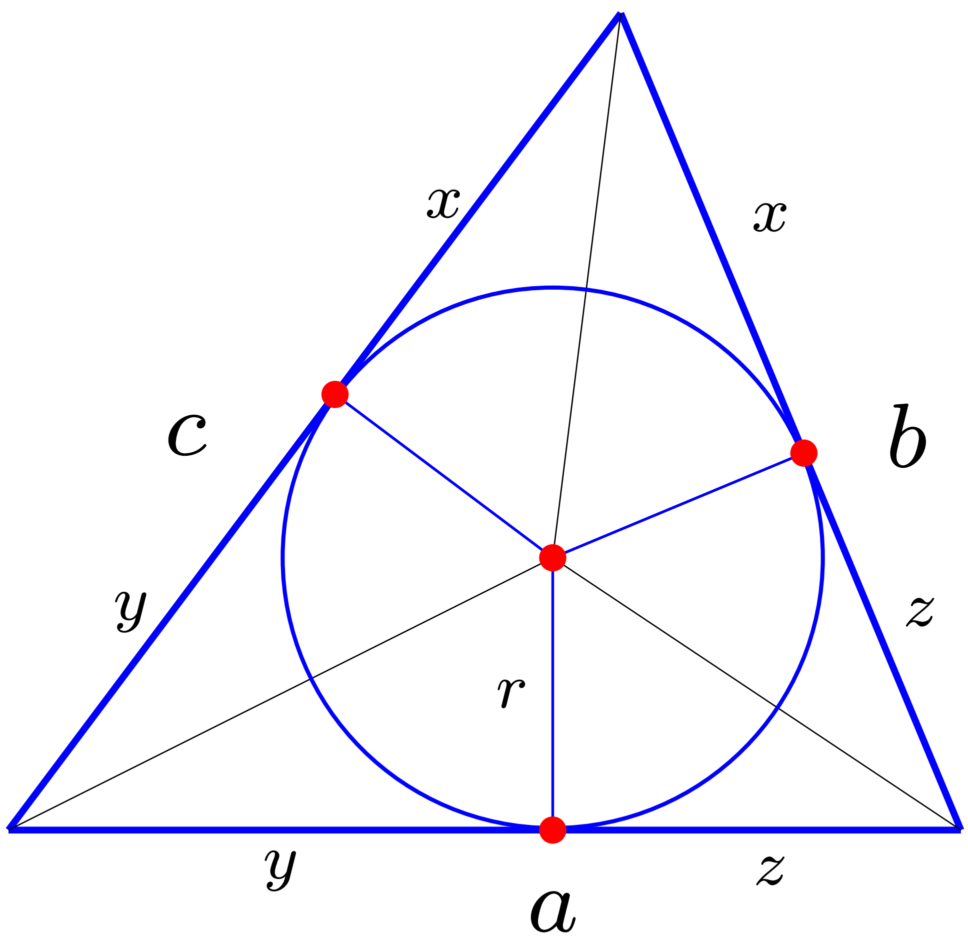
New variables



$$s = \frac{1}{2}(a + b + c)$$
$$s = x + y + z$$
$$x = s - a$$
$$y = s - b$$
$$z = s - c$$

$$A = rs$$

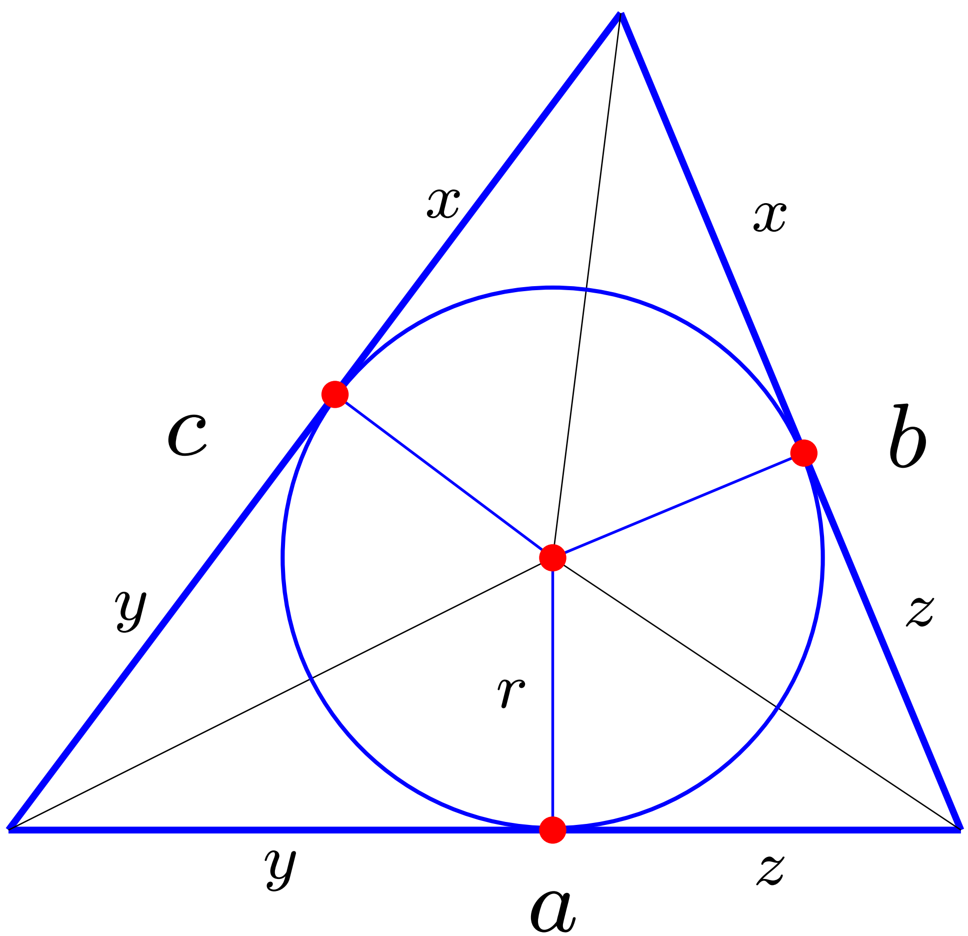
New variables



$$s = \tfrac{1}{2}(a + b + c)$$
$$s = x + y + z$$
$$x = s - a$$
$$y = s - b$$
$$z = s - c$$

$$A = rs$$
$$A^2 = sxyz$$

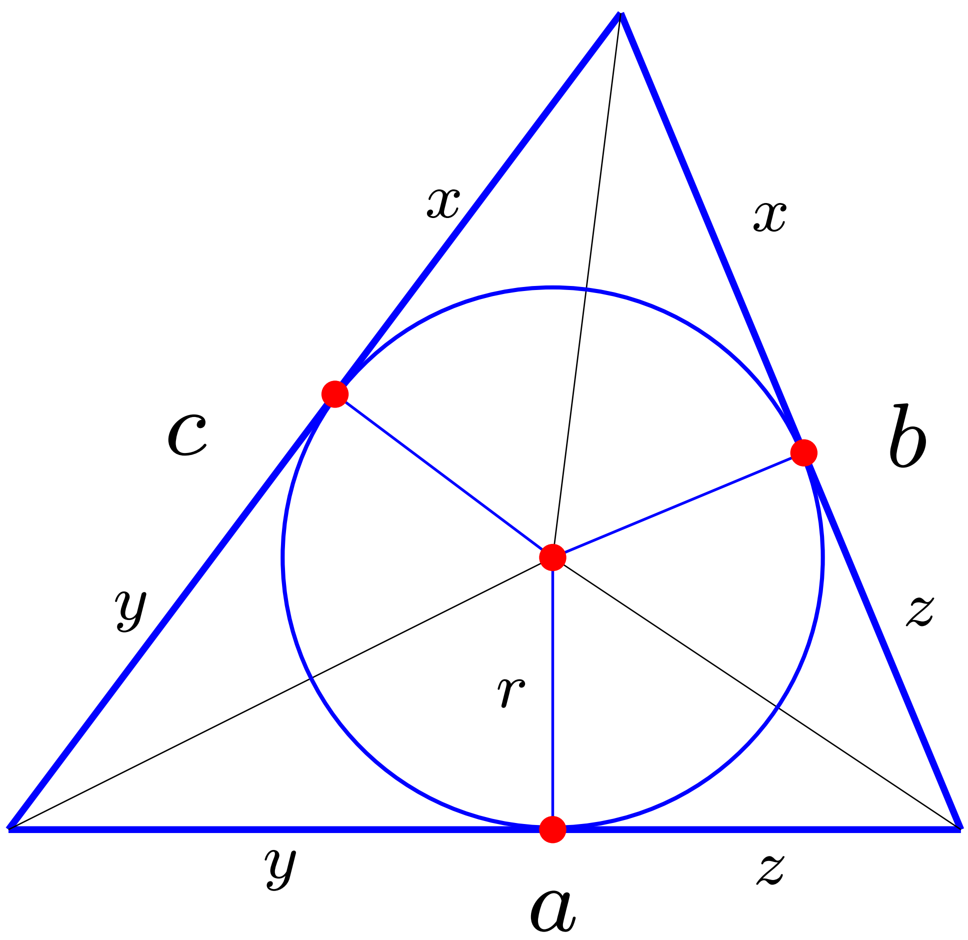
New variables



$$s = \frac{1}{2}(a+b+c)$$
$$s = x+y+z$$
$$x = s-a$$
$$y = s-b$$
$$z = s-c$$

$$A = rs$$
$$A^2 = sxyz$$
$$r^2s^2 = sxyz$$

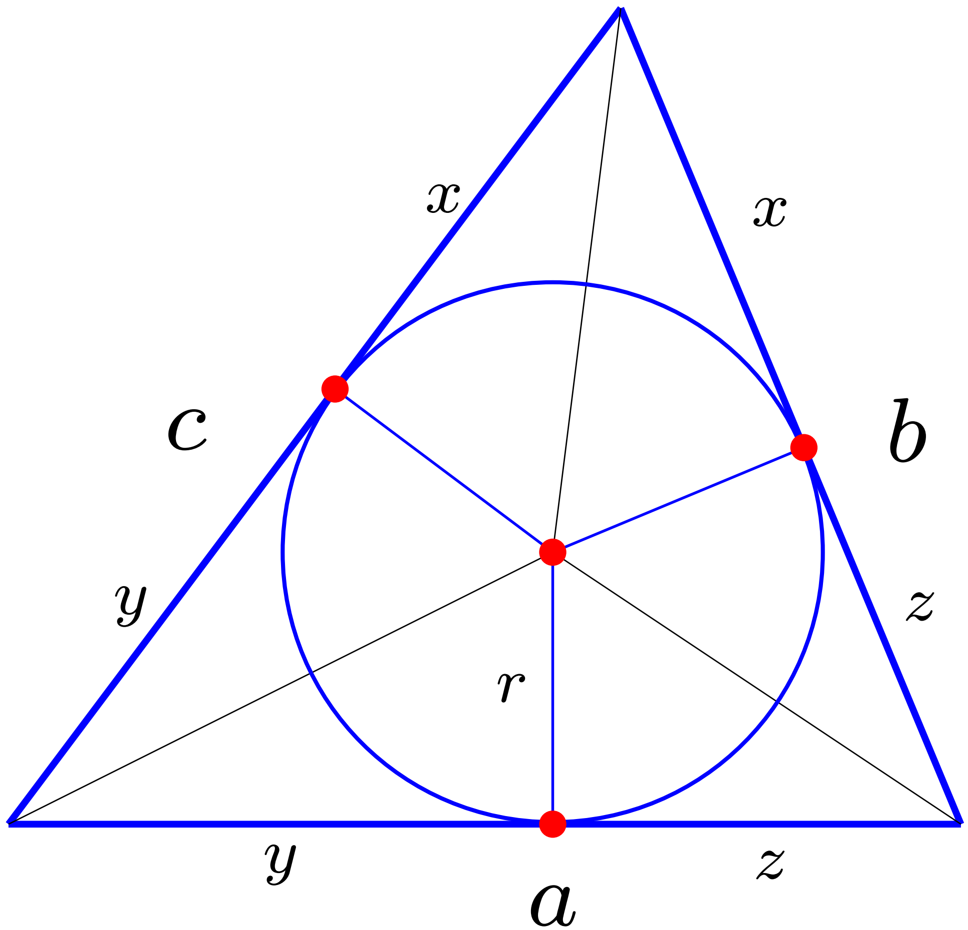
New variables



$$s = \frac{1}{2}(a+b+c)$$
$$s = x+y+z$$
$$x = s-a$$
$$y = s-b$$
$$z = s-c$$

$$A = rs$$
$$A^2 = sxyz$$
$$r^2s^2 = sxyz$$
$$r^2(x+y+z) = xyz$$

New variables



$s = \frac{1}{2}(a + b + c)$
$s = x + y + z$
$x = s - a$
$y = s - b$
$z = s - c$

$A = rs$
$A^2 = sxyz$
$r^2 s^2 = sxyz$
$r^2(x + y + z) = xyz$

We want constant $\frac{r}{x+y+z}$

Every triangle gives $x, y, z, r > 0$ satisfying

$$r^2(x + y + z) = xyz.$$

Triangles with the same area and perimeter yield the same ratio

$$t = \frac{r}{x + y + z}.$$

Every triangle gives $x, y, z, r > 0$ satisfying

$$r^2(x + y + z) = xyz.$$

Triangles with the same area and perimeter yield the same ratio

$$t = \frac{r}{x + y + z}.$$

For fixed parameter t we want solutions $x, y, z > 0$ to

$$t^2(x + y + z)^3 = xyz.$$

Every triangle gives $x, y, z, r > 0$ satisfying

$$r^2(x+y+z) = xyz.$$

Triangles with the same area and perimeter yield the same ratio

$$t = \frac{r}{x+y+z}.$$

For fixed parameter t we want solutions $x, y, z > 0$ to

$$t^2(x+y+z)^3 = xyz.$$

We can find solutions for t of the form

$$t = \frac{s-1}{s(s+1)}.$$

Theorem 1. *There exists a sequence* $\{(x_n, y_n, z_n)\}_{n\geq 1}$ *of triples of elements in* $\mathbb{Q}(s)$ *such that*

1. *for all* $n \geq 1$ *and all* $\sigma \in \mathbb{R}$ *with* $\sigma > 1$*, there exists a triangle* $\Delta_n(\sigma)$ *with sides* $y_n(\sigma) + z_n(\sigma)$*,* $x_n(\sigma) + z_n(\sigma)$*, and* $x_n(\sigma) + y_n(\sigma)$*, and inradius* $(\sigma - 1)\sigma^{-1}(\sigma + 1)^{-1}(x_n(\sigma) + y_n(\sigma) + z_n(\sigma))$*, and*

2. *for all* $m, n \geq 1$ *and* $\sigma_0, \sigma_1 \in \mathbb{Q}$ *with* $\sigma_0, \sigma_1 > 1$*, the rational triangles* $\Delta_m(\sigma_0)$ *and* $\Delta_n(\sigma_1)$ *are similar if and only if* $m = n$ *and* $\sigma_0 = \sigma_1$*.*

First triples in the sequence:

$$(x_1, y_1, z_1) = \big(1+s, -1+s, (-1+s)s\big),$$

$$x_2 = (-1+s)(1+6s-2s^2-2s^3+s^4)^3,$$

$$y_2 = (-1+s)(-1+4s+4s^2-4s^3+s^4)^3,$$

$$z_2 = s(1+s)(3+4s^2-4s^3+s^4)^3,$$

$$x_3 = (-1+s)(1+2s+2s^2-2s^3+s^4)^3 (-1-22s+66s^2+14s^3-72s^4+30s^5+6s^6-6s^7+s^8)^3,$$

$$y_3 = (1+s)(-1+20s+68s^2-84s^3+139s^4+32s^5-224s^6+ 64s^7+149s^8-148s^9+60s^{10}-12s^{11}+s^{12})^3,$$

$$z_3 = (-1+s)s(5+10s+126s^2+62s^3-225s^4+52s^5+28s^6+ 12s^7+27s^8-62s^9+38s^{10}-10s^{11}+s^{12})^3,$$

$$x_4 = (1+s)(-1-62s+198s^2+1698s^3+7764s^4-8298s^5-10830s^6+43622s^7-15685s^8 -45356s^9-1348s^{10}+75284s^{11}-13088s^{12}-93076s^{13}+85220s^{14}+12s^{15}-49467s^{16} +40842s^{17}-16034s^{18}+2282s^{19}+844s^{20}-546s^{21}+138s^{22}-18s^{23}+s^{24})^3,$$

$$y_4 = (-1+s)(-1+54s+550s^2-10s^3+5092s^4+16674s^5+98s^6-51662s^7+22875s^8+ 41916s^9-63076s^{10}+45628s^{11}+13088s^{12}-63644s^{13}+38884s^{14}+17668s^{15}- 31195s^{16}+8302s^{17}+8990s^{18}-9554s^{19}+4476s^{20}-1254s^{21}+218s^{22}-22s^{23}+s^{24})^3,$$

$$z_4 = (-1+s)s(-7-28s-1168s^2-2588s^3+5170s^4+6940s^5+20176s^6-10628s^7- 70305s^8+46664s^9+85440s^{10}-107832s^{11}+380s^{12}+66840s^{13}-46848s^{14}+13656s^{15}- 1465s^{16}-2796s^{17}+5712s^{18}-5228s^{19}+2738s^{20}-884s^{21}+176s^{22}-20s^{23}+s^{24})^3.$$

Sketch of a very unenlightening proof:

Step 1. Show there is one solution to

$$\frac{(s-1)^2}{s^2(s+1)^2}(x+y+z)^3 = xyz.$$

Sketch of a very unenlightening proof:

Step 1. Show there is one solution to

$$\frac{(s-1)^2}{s^2(s+1)^2}(x+y+z)^3 = xyz.$$

A solution is $(x,y,z) = (s+1, s-1, s(s-1))$ (positive for $s > 1$).

Sketch of a very unenlightening proof:

Step 1. Show there is one solution to

$$\frac{(s-1)^2}{s^2(s+1)^2}(x+y+z)^3 = xyz.$$

A solution is $(x, y, z) = (s+1, s-1, s(s-1))$ (positive for $s > 1$).

Step 2. Show that if there is a solution, then there are infinitely many solutions.

If (x, y, z) is a solution, then so is (x', y', z') with ...

$$
\begin{aligned}
x' =& -3(s+1)(2s^2-s+1)(s-1)^3z^3 - 2(s+1)(4s^3-10s^2+7s-3)(s-1)^2xz^2\\
&+3(2s^6-s^4+3s^3+5s^2-3s+2)(s-1)^2yz^2 - 3(s+1)(-1+s)^5x^2z\\
&+3s(s^4+s^2+2)(s-1)^3y^3 + s(s^4-s^3+s^2-3s+6)(s-1)^3x^2y\\
&-(s-1)(s^9+s^8-6s^7+15s^6-6s^5+6s^4-14s^3+31s^2-15s+3)y^2z\\
&+3s(s^4-s^3+s^2-s+4)(s-1)^3xy^2 + 3(s^2+2s+2)(s-1)^6xyz,\\
y' =& -3(s+1)(s^4-s^3+4s^2-s+1)(s-1)^2z^3 + 3s(s^2+s+2)(s-1)^4y^3\\
&+(s^9-s^8+4s^7+6s^6-3s^5+9s^4+46s^3-40s^2+16s-6)yz^2 + 6s(s-1)^4x^2y\\
&-(s+1)(s^4-4s^3+10s^2-6s+3)(s-1)^2x^2z - 6(5s^2-2s+1)(s-1)^2xyz\\
&-3(s^6+4s^4+3s^3+10s^2-3s+1)(s-1)^2y^2z - s(s^3-2s^2-3s-12)(s-1)^4xy^2\\
&-3(s+1)(s^4-3s^3+7s^2-3s+2)(s-1)^2xz^2,\\
z' =& -s(-1+s)(s+1)^4z^3 + 3s^2(-1+s)^2(s+1)^3yz^2 + s^4(s+1)(-1+s)^4y^3\\
&-3s^3(s+1)^2(-1+s)^3y^2z.
\end{aligned}
$$

Problem 1: Even if $x, y, z > 0$, we may not have $x', y', z' > 0$.

Problem 1: Even if $x, y, z > 0$, we may not have $x', y', z' > 0$.

Solution: Do the same operation twice (ouch) ...

Problem 1: Even if $x, y, z > 0$, we may not have $x', y', z' > 0$.

Solution: Do the same operation twice (ouch) ...

Problem 2: Do we get infinitely many?

Problem 3: Are the triangles nonsimilar?

Problem 1: Even if $x, y, z > 0$, we may not have $x', y', z' > 0$.

Solution: Do the same operation twice (ouch) ...

Problem 2: Do we get infinitely many?

Problem 3: Are the triangles nonsimilar?

Solution: We need an interpretation of these formulas.

The change of variables

$$\begin{aligned} x &= -s(s+1)p + q \\ y &= -s(s+1)p - q \\ z &= 8s(s+1)(s-1)^2 \end{aligned}$$

shows that our curve in $\mathbb{P}^2$ with parameter s given by

$$(s-1)^2(x+y+z)^3 = s^2(s+1)^2 xyz$$

is isomorphic to the curve given by

$$q^2 = (p - 4(s-1)^2)^3 + s^2(s+1)^2 p^2$$

together with the point at infinity.

Interlude on elliptic curves

Definition:
An **elliptic curve** is a curve given by an equation of the form

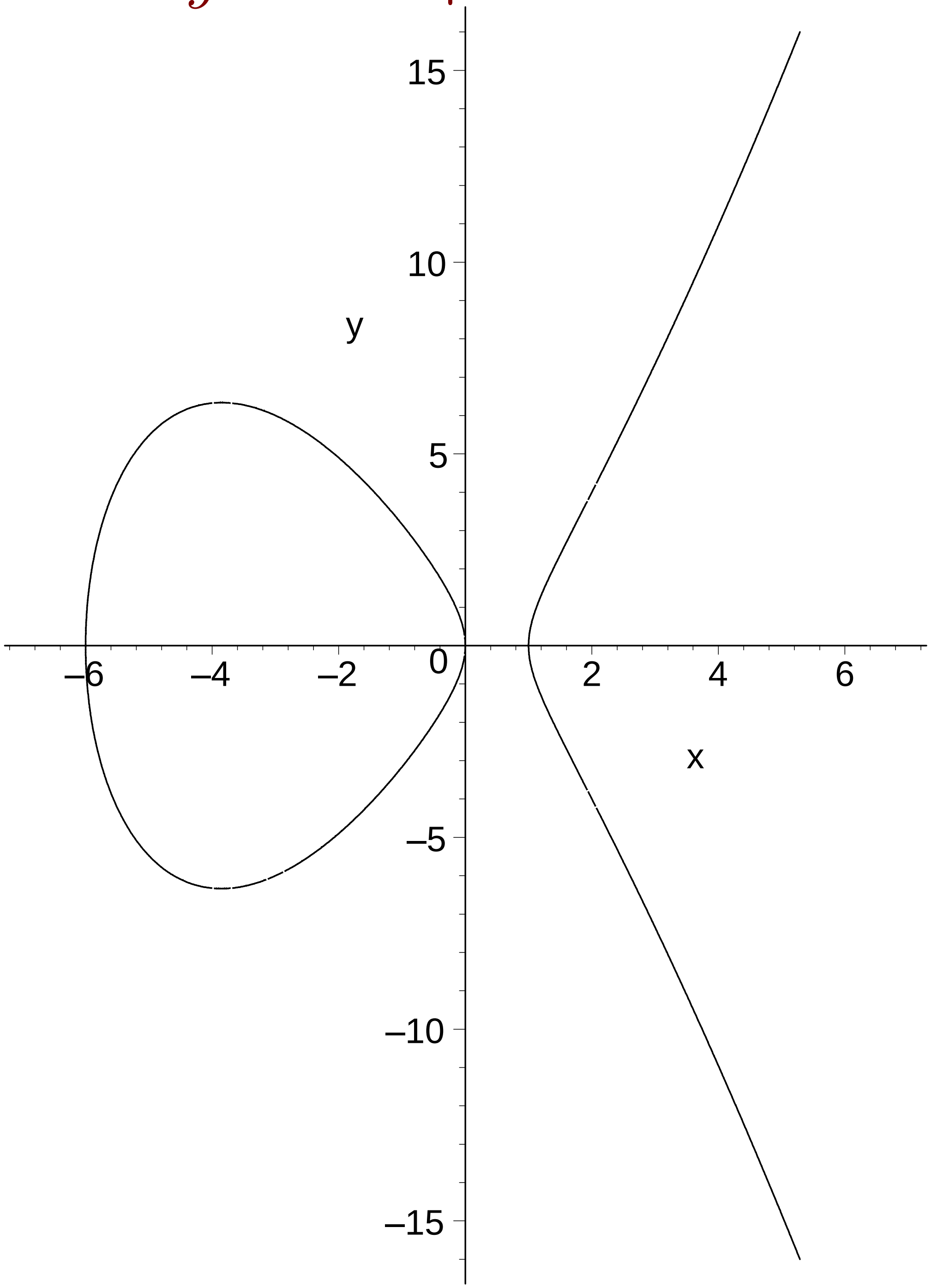
$$y^2 = f(x)$$

with f a separable polynomial of degree 3, together with the point at infinity.

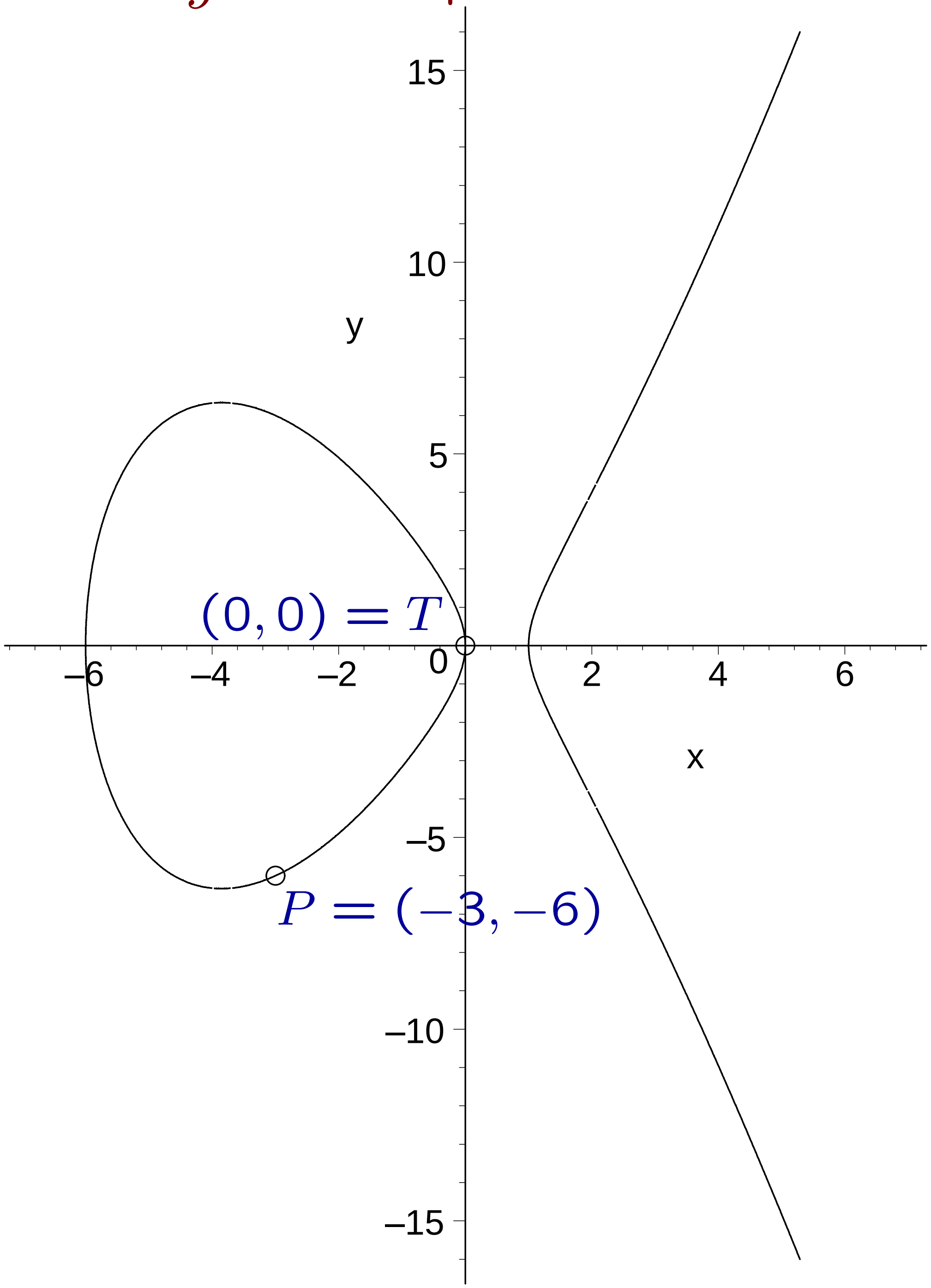
Elliptic curves are

- 1-dimensional lie-groups,
- Calabi-Yau 1-manifolds,
- 1-dimensional abelian varieties.

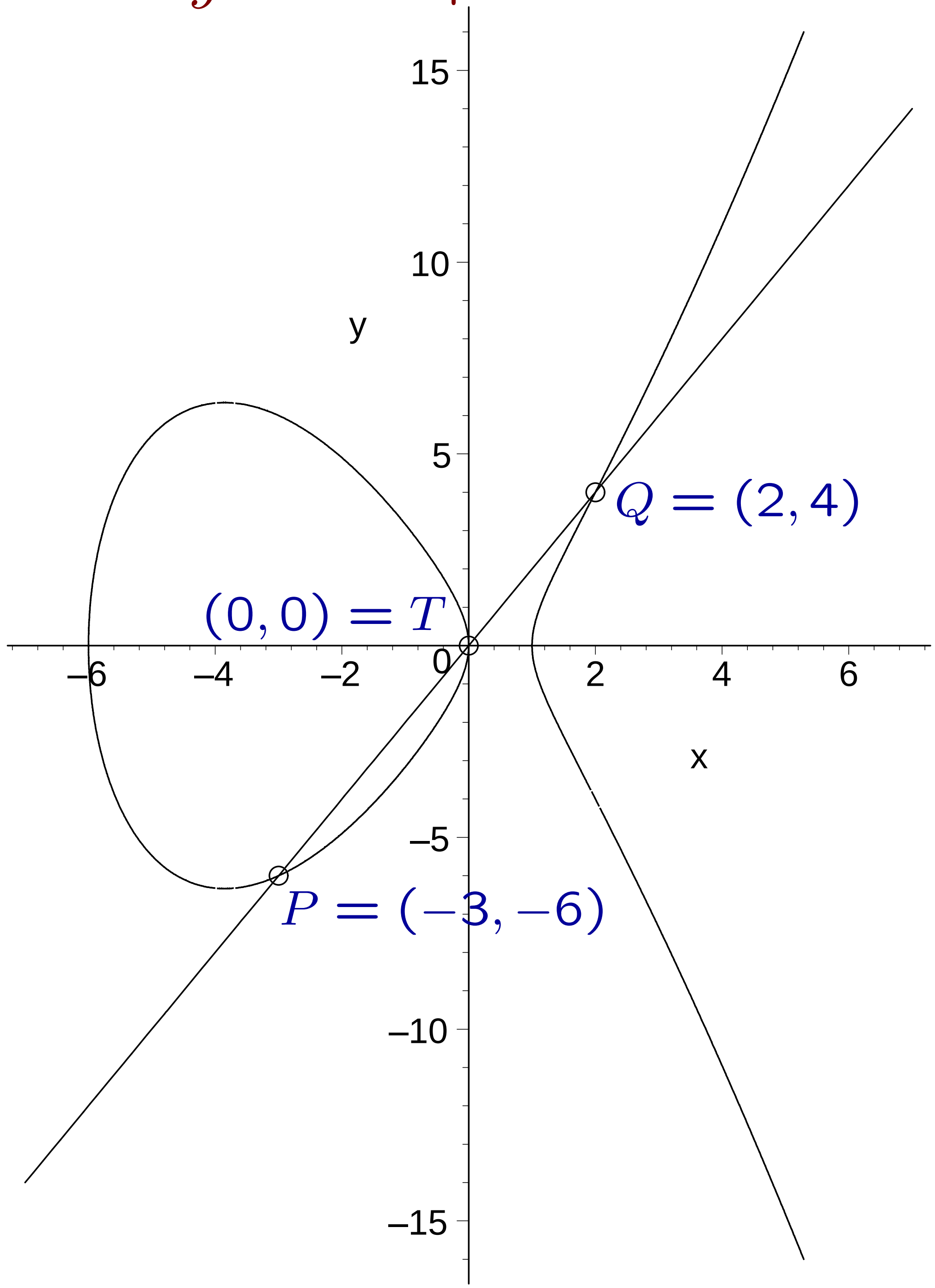
$$y^2 = x^3 + 5x^2 - 6x$$



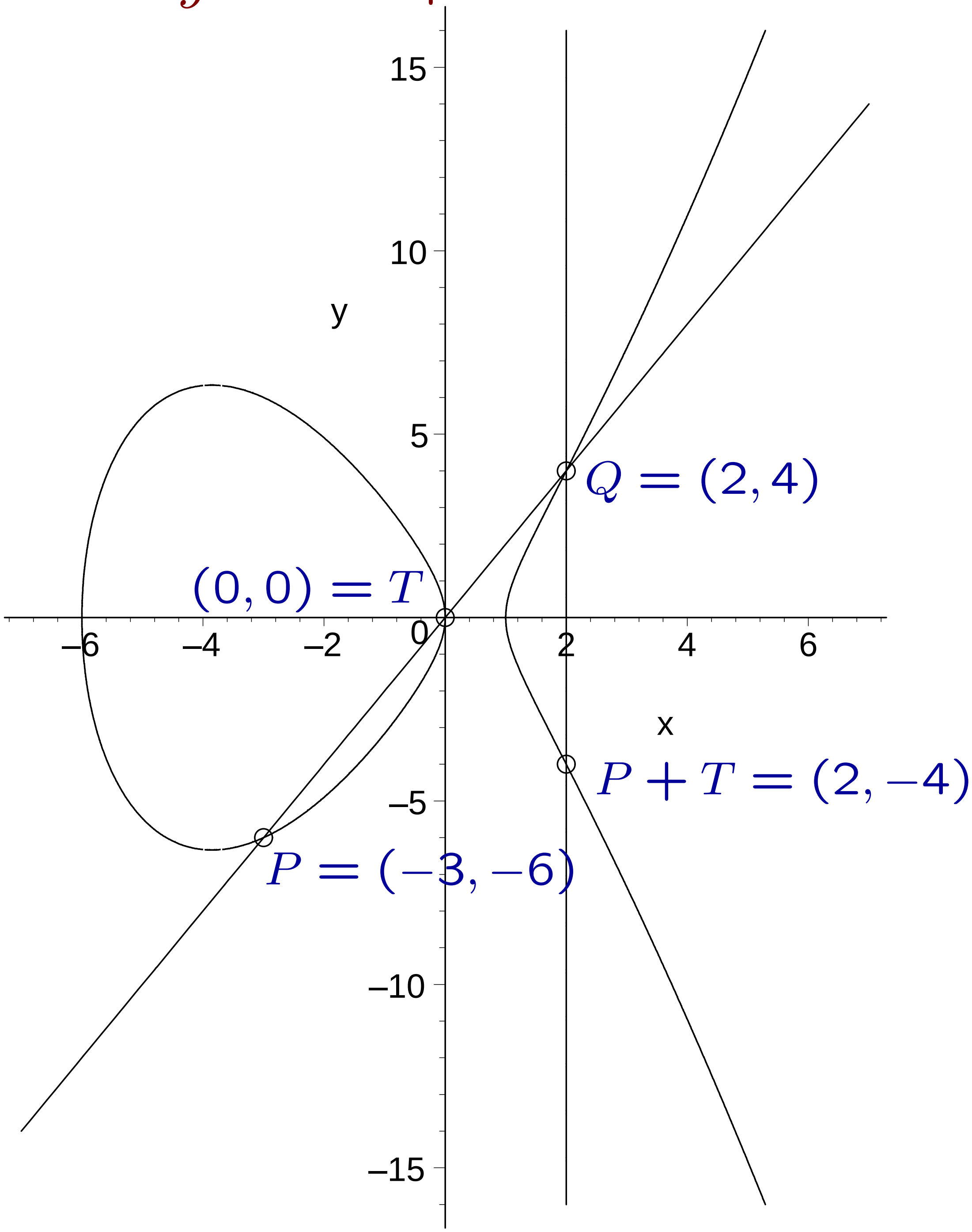
$$y^2 = x^3 + 5x^2 - 6x$$



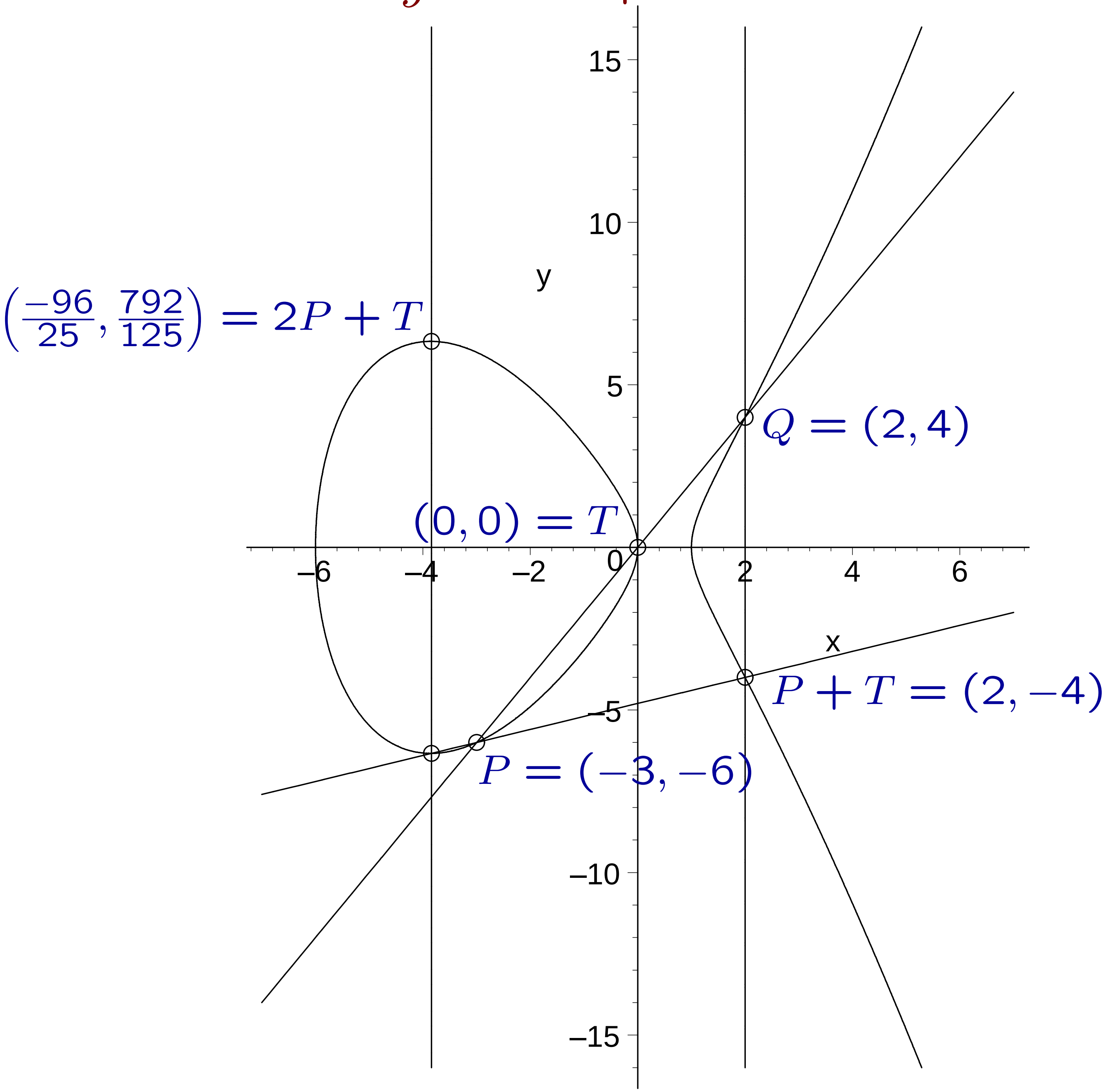
$$y^2 = x^3 + 5x^2 - 6x$$



$$y^2 = x^3 + 5x^2 - 6x$$



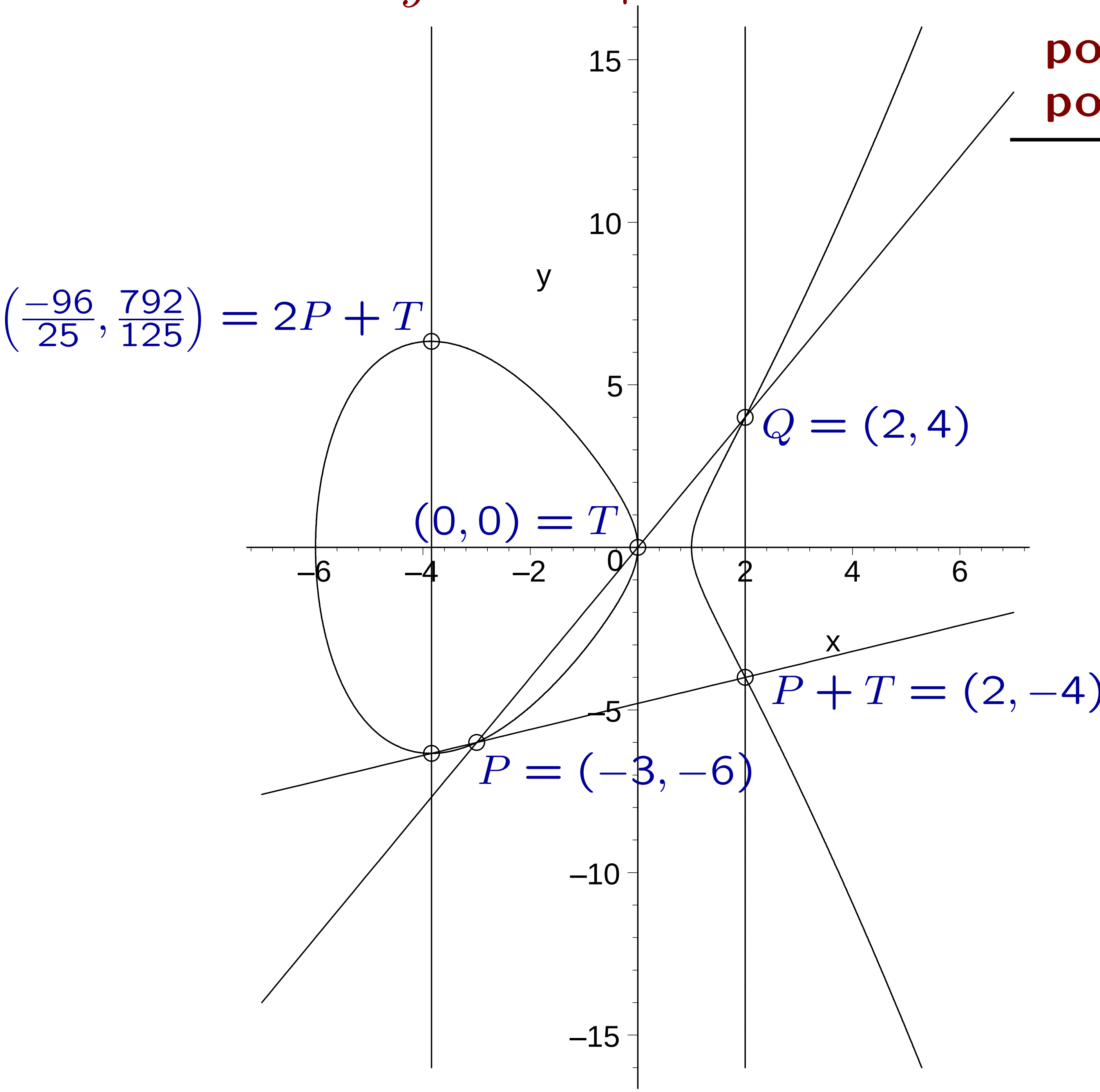
$$y^2 = x^3 + 5x^2 - 6x$$



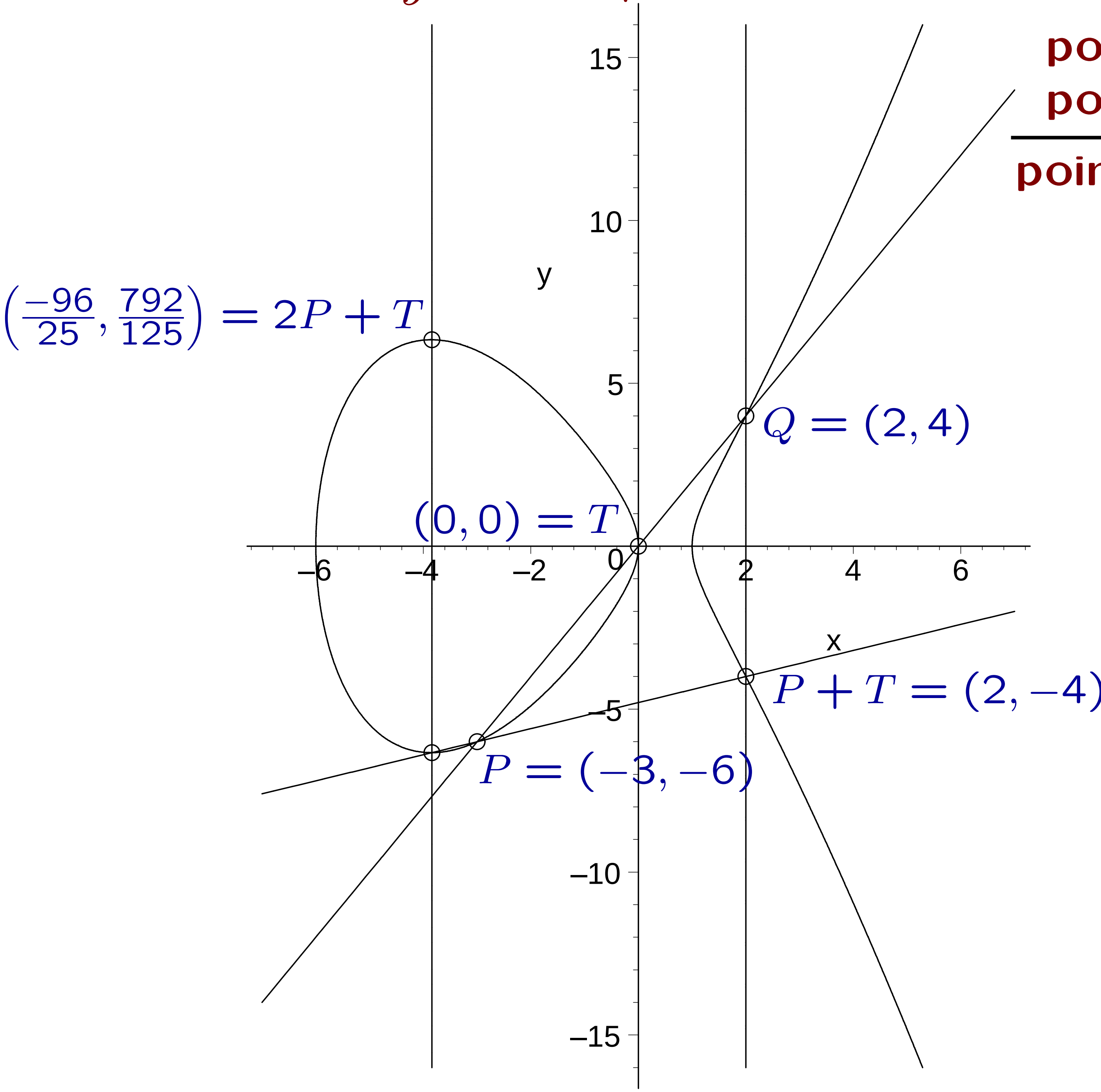
15
10
5
0
–5
–10
–15
–6
–4
–2
2
4
6
y
x
$\left(\frac{-96}{25}, \frac{792}{125}\right) = 2P + T$
$Q = (2, 4)$
$(0, 0) = T$
$P + T = (2, -4)$
$P = (-3, -6)$

$$y^2 = x^3 + 5x^2 - 6x$$

point ON egg
point ON egg +



$$y^2 = x^3 + 5x^2 - 6x$$

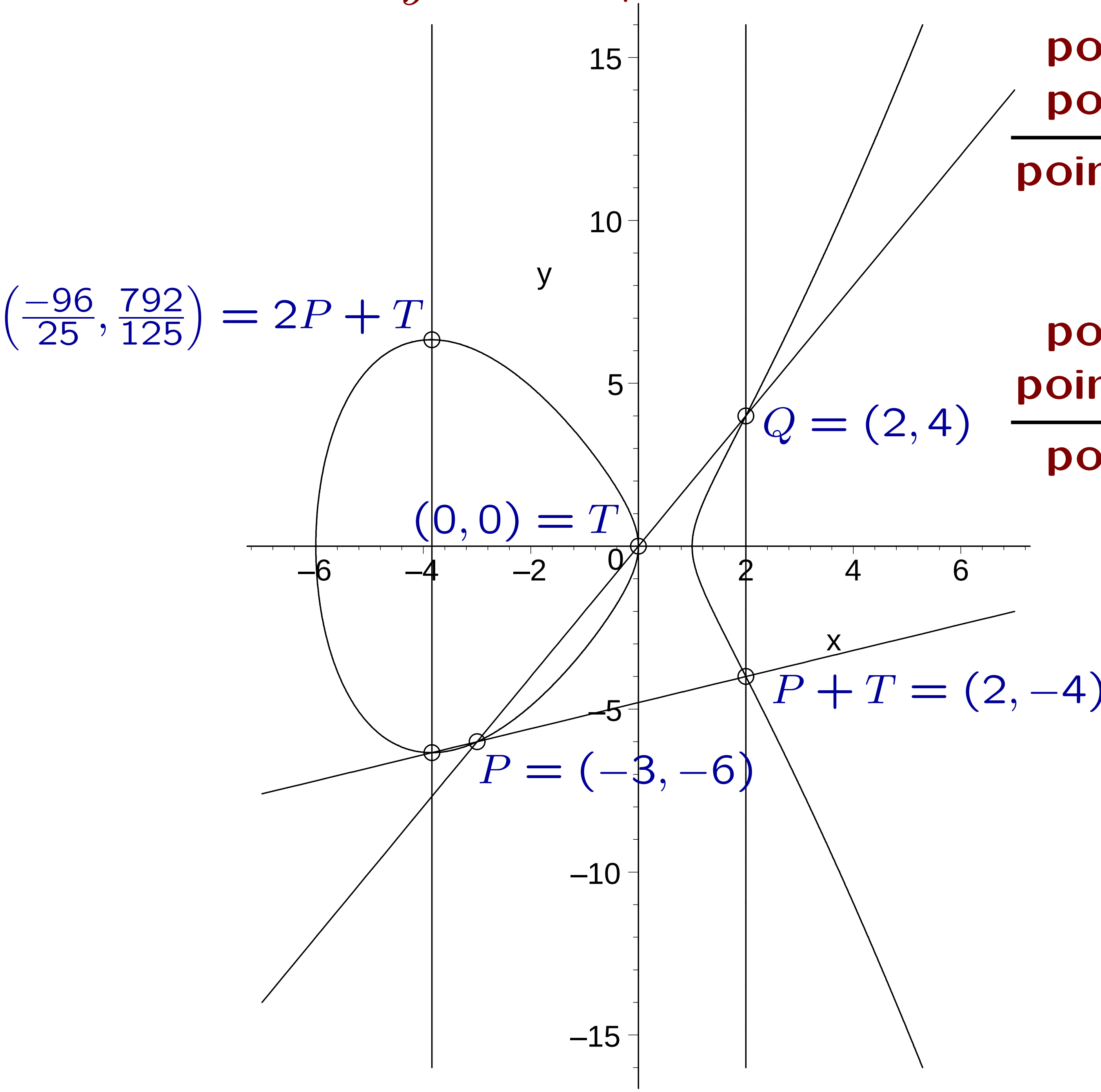


point ON egg
point ON egg +

---

point OFF egg

$$y^2 = x^3 + 5x^2 - 6x$$



$$\begin{array}{r} \textbf{point ON egg} \\ \textbf{point ON egg} + \\ \hline \textbf{point OFF egg} \end{array}$$

$$\begin{array}{r} \textbf{point ON egg} \\ \textbf{point OFF egg} + \\ \hline \textbf{point ON egg} \end{array}$$

Back to our curve

$$(s-1)^2(x+y+z)^3 = s^2(s+1)^2xyz$$

and the isomorphism

$$x = -s(s+1)p + q$$
$$y = -s(s+1)p - q$$
$$z = 8s(s+1)(s-1)^2$$

to the curve given by

$$q^2 = (p - 4(s-1)^2)^3 + s^2(s+1)^2p^2.$$

Back to our curve

$$(s-1)^2(x+y+z)^3 = s^2(s+1)^2xyz$$

and the isomorphism

$$\begin{aligned} x &= -s(s+1)p + q \\ y &= -s(s+1)p - q \\ z &= 8s(s+1)(s-1)^2 \end{aligned}$$

to the curve given by

$$q^2 = (p - 4(s-1)^2)^3 + s^2(s+1)^2p^2.$$

For $s > 1$ the inequalities $x, y, z > 0$ are equivalent with

$$p < 0, \qquad \text{and} \qquad s^2(s+1)^2p^2 > q^2 = (p-4(s-1)^2)^3 + s^2(s+1)^2p^2,$$

so just to $p < 0$. This turns out to be equivalent to

"lying on the egg-part of the elliptic curve."

Our simple solution $(x, y, z) = (s + 1, s - 1, s(s - 1))$ corresponds to the point $R = (8 - 8s, 8s^2 - 8)$ on the egg-part of the elliptic curve for $s > 1$.

The point $P = (4(s - 1)^2, 4s(s + 1)(s - 1)^2)$ has order 3.

Our simple solution $(x, y, z) = (s+1, s-1, s(s-1))$ corresponds to the point $R = (8-8s, 8s^2-8)$ on the egg-part of the elliptic curve for $s > 1$.

The point $P = (4(s-1)^2, 4s(s+1)(s-1)^2)$ has order 3.

(1) R has infinite order for rational s (Mazur's Theorem),
(2) All odd multiples of R lie on the egg-part,
(3) mR and nR give similar triangles iff $mR = \pm nR + kP$ for some k,

Our simple solution $(x, y, z) = (s+1, s-1, s(s-1))$ corresponds to the point $R = (8-8s, 8s^2-8)$ on the egg-part of the elliptic curve for $s > 1$.

The point $P = (4(s-1)^2, 4s(s+1)(s-1)^2)$ has order 3.

(1) R has infinite order for rational s (Mazur's Theorem),
(2) All odd multiples of R lie on the egg-part,
(3) mR and nR give similar triangles iff $mR = \pm nR + kP$ for some k,

We conclude that the positive odd multiples of R give our wanted parametrizations.

Our simple solution $(x, y, z) = (s + 1, s - 1, s(s - 1))$ corresponds to the point $R = (8 - 8s, 8s^2 - 8)$ on the egg-part of the elliptic curve for $s > 1$.

The point $P = (4(s - 1)^2, 4s(s + 1)(s - 1)^2)$ has order 3.

(1) R has infinite order for rational s (Mazur's Theorem),
(2) All odd multiples of R lie on the egg-part,
(3) mR and nR give similar triangles iff $mR = \pm nR + kP$ for some k,

We conclude that the positive odd multiples of R give our wanted parametrizations.

(4) big formula (x', y', z') corresponds to $Q \mapsto Q + R$.

Unsatisfying:

Unsatisfying:

- Where did the substitution

$$t = \frac{s-1}{s(s+1)}$$

  come from?

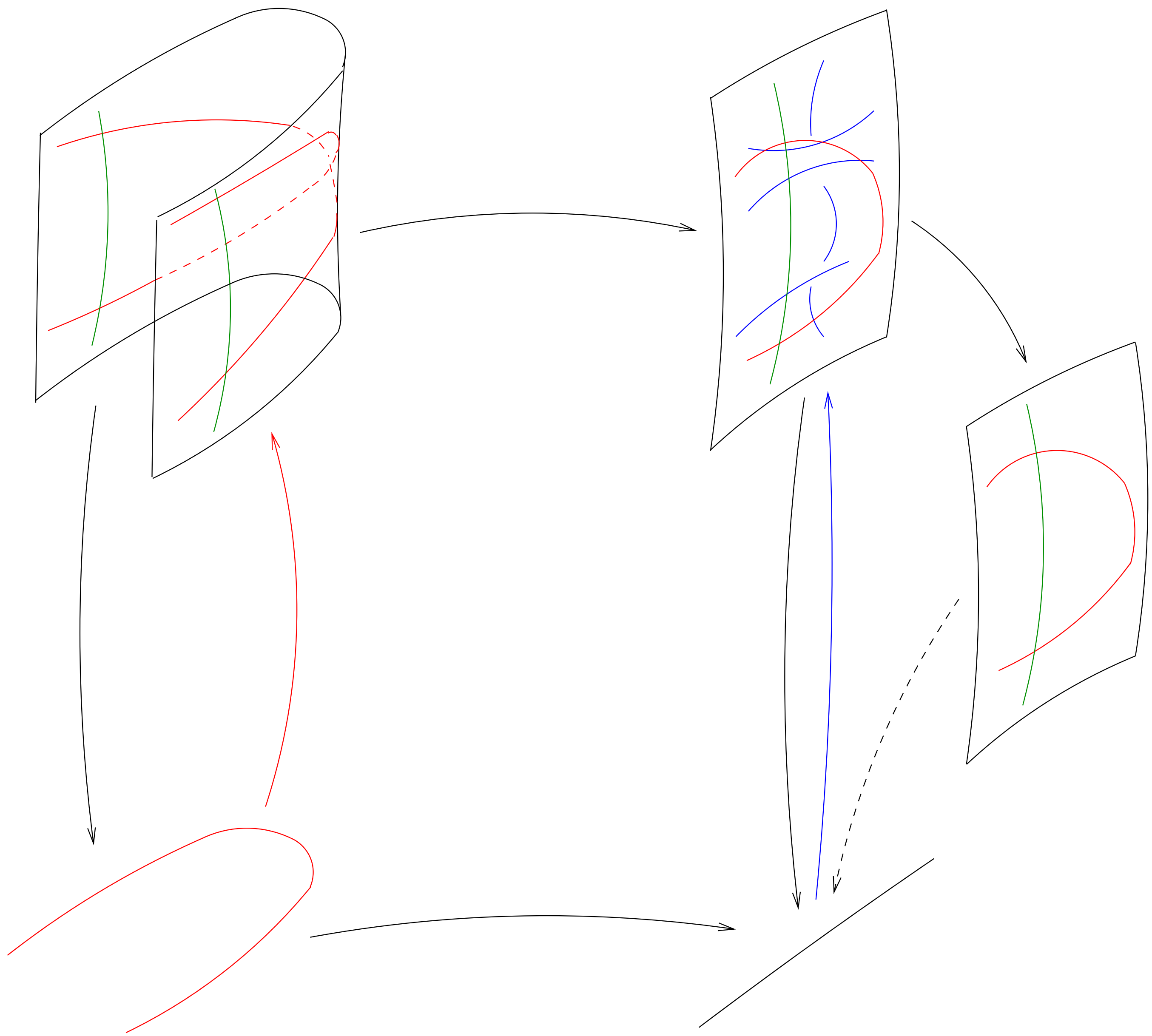
Unsatisfying:

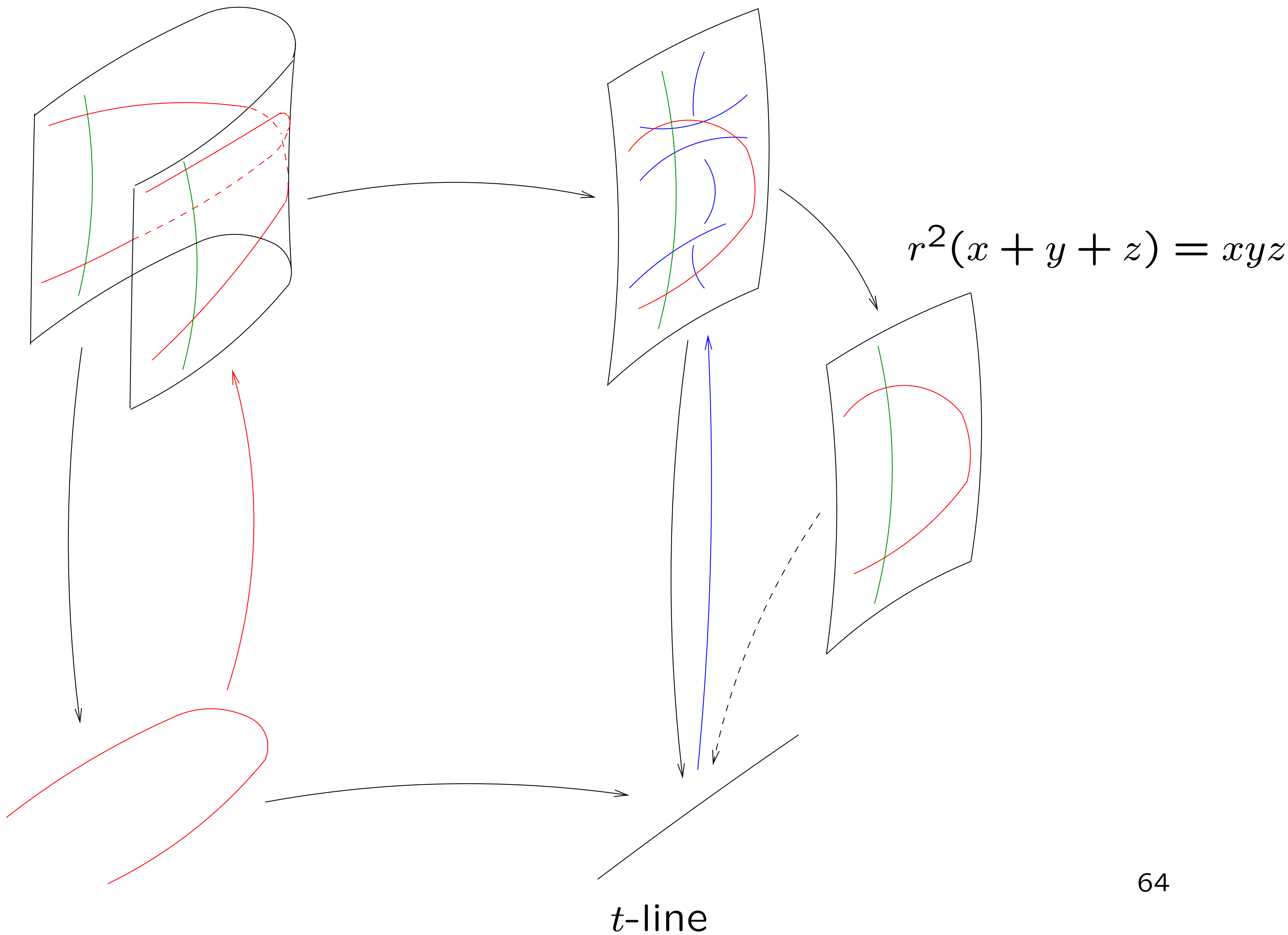
- Where did the substitution

$$t = \frac{s-1}{s(s+1)}$$

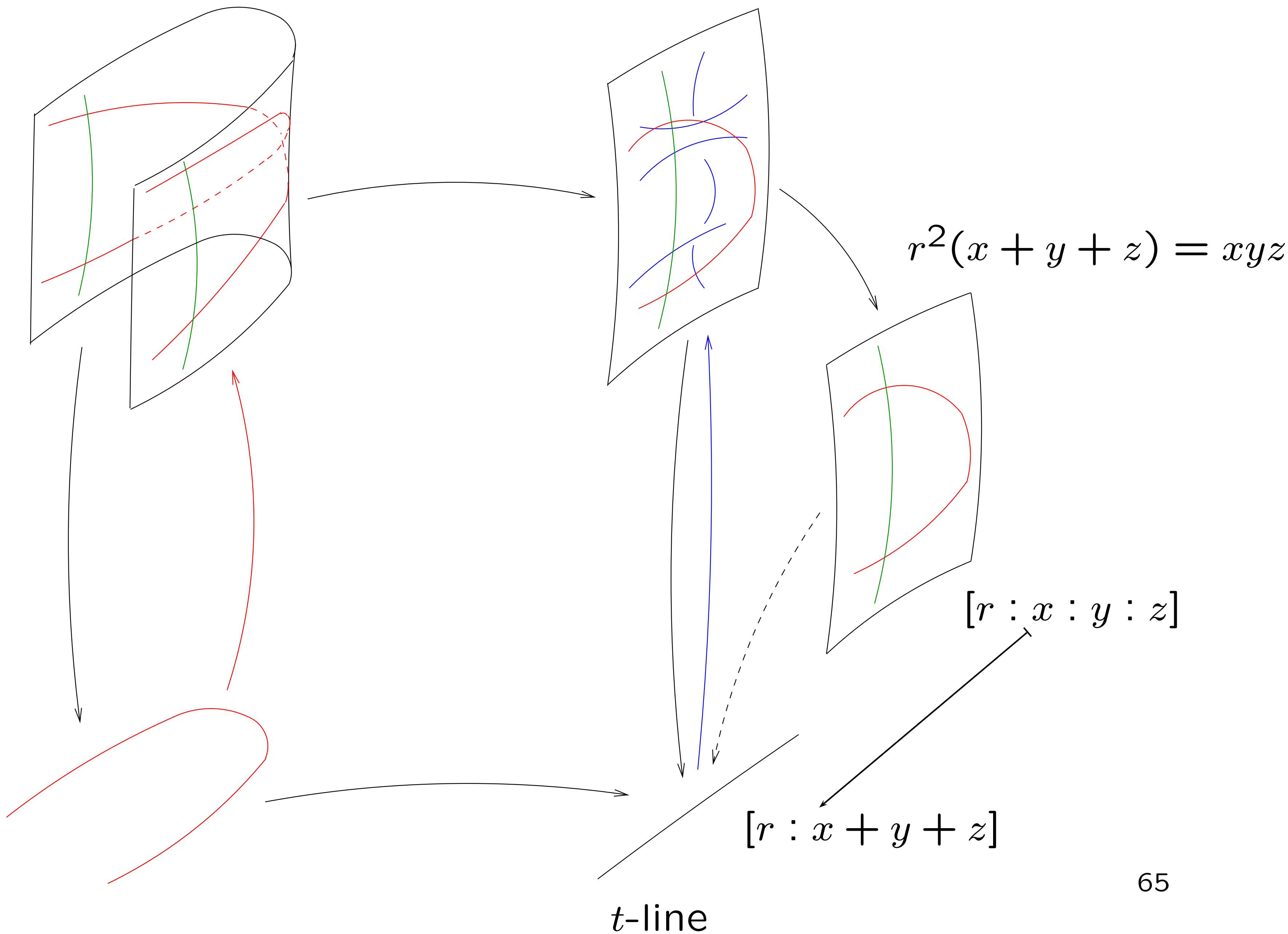
  come from?

- Where are the surfaces from the title of this talk?

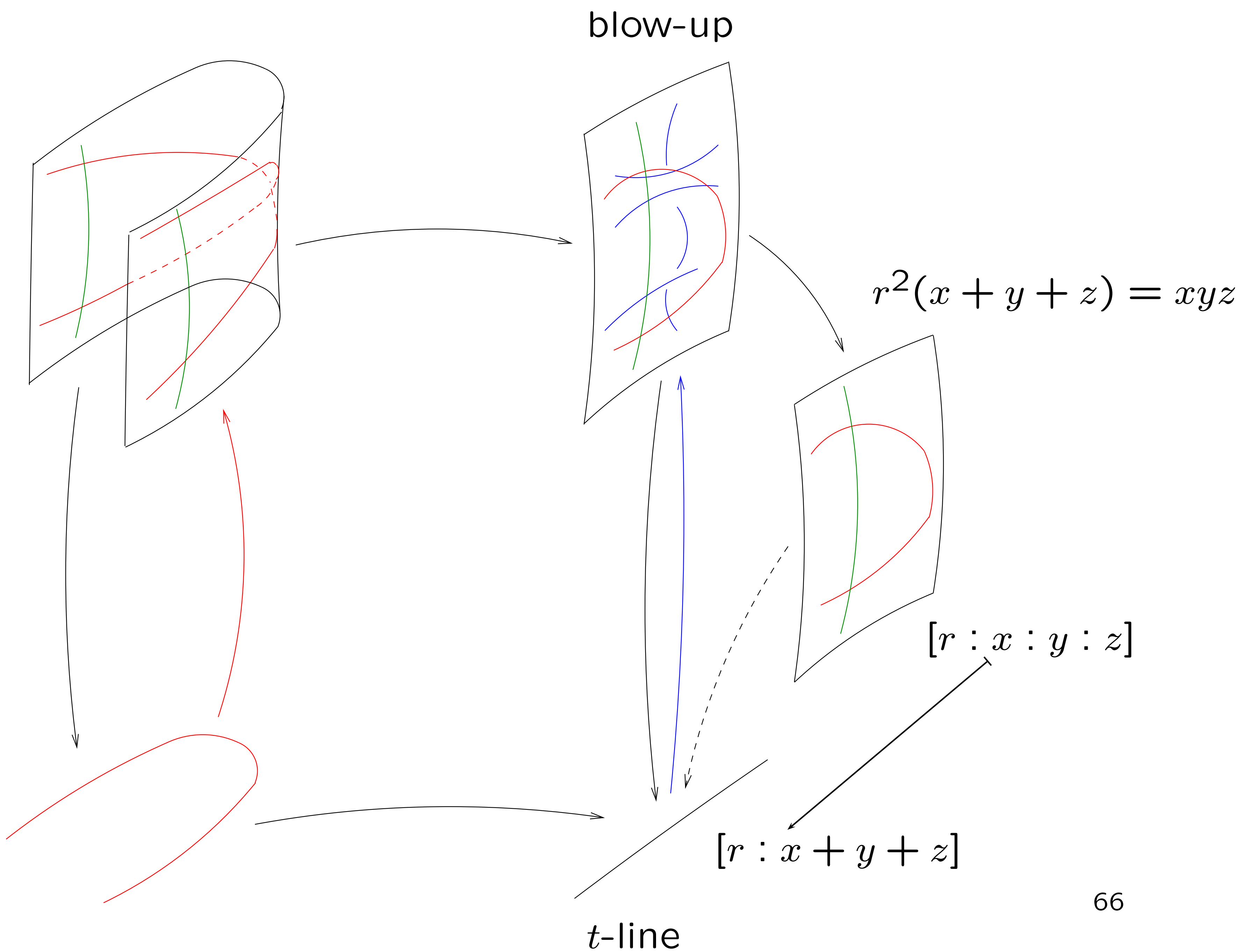


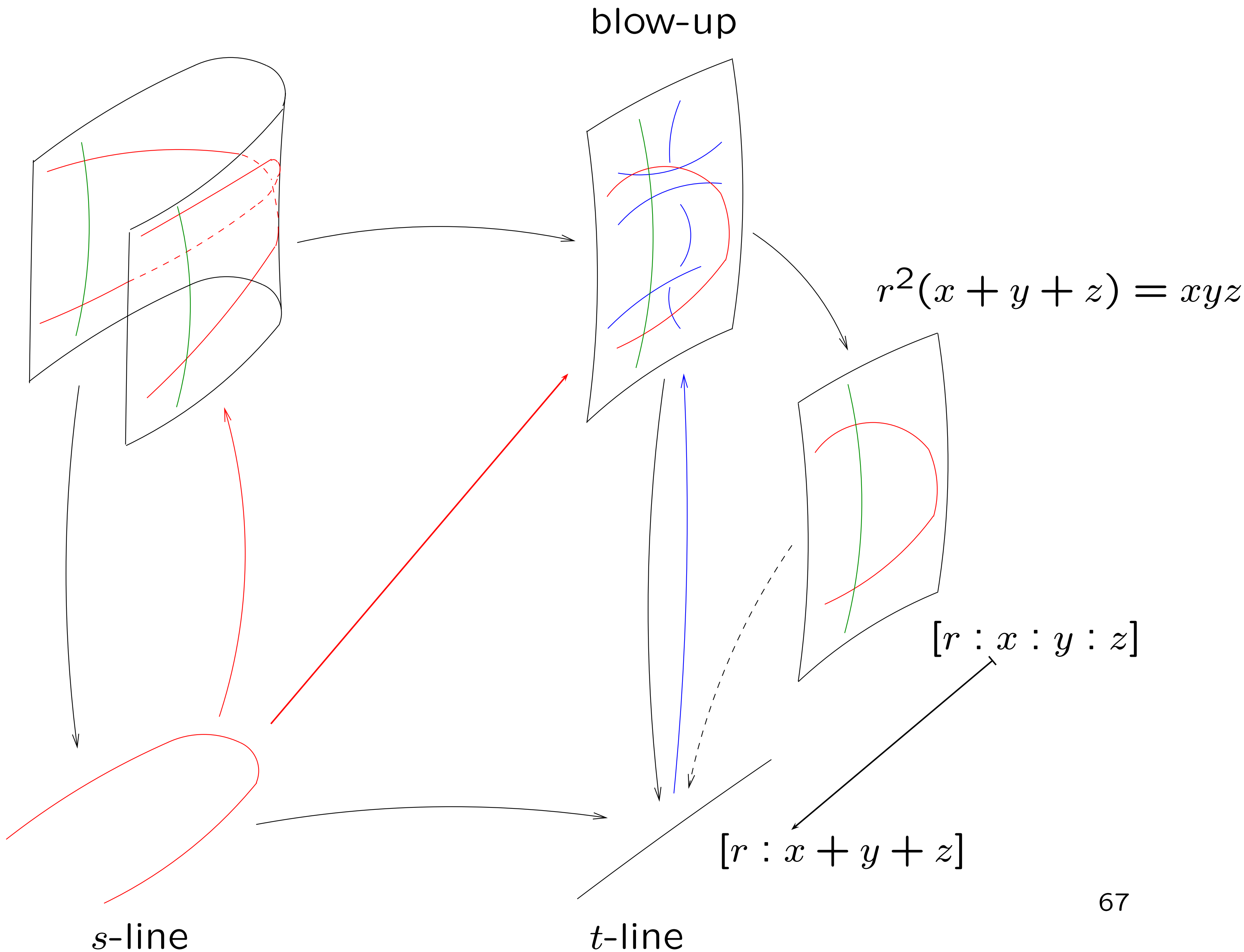


$r^2(x+y+z) = xyz$
t-line

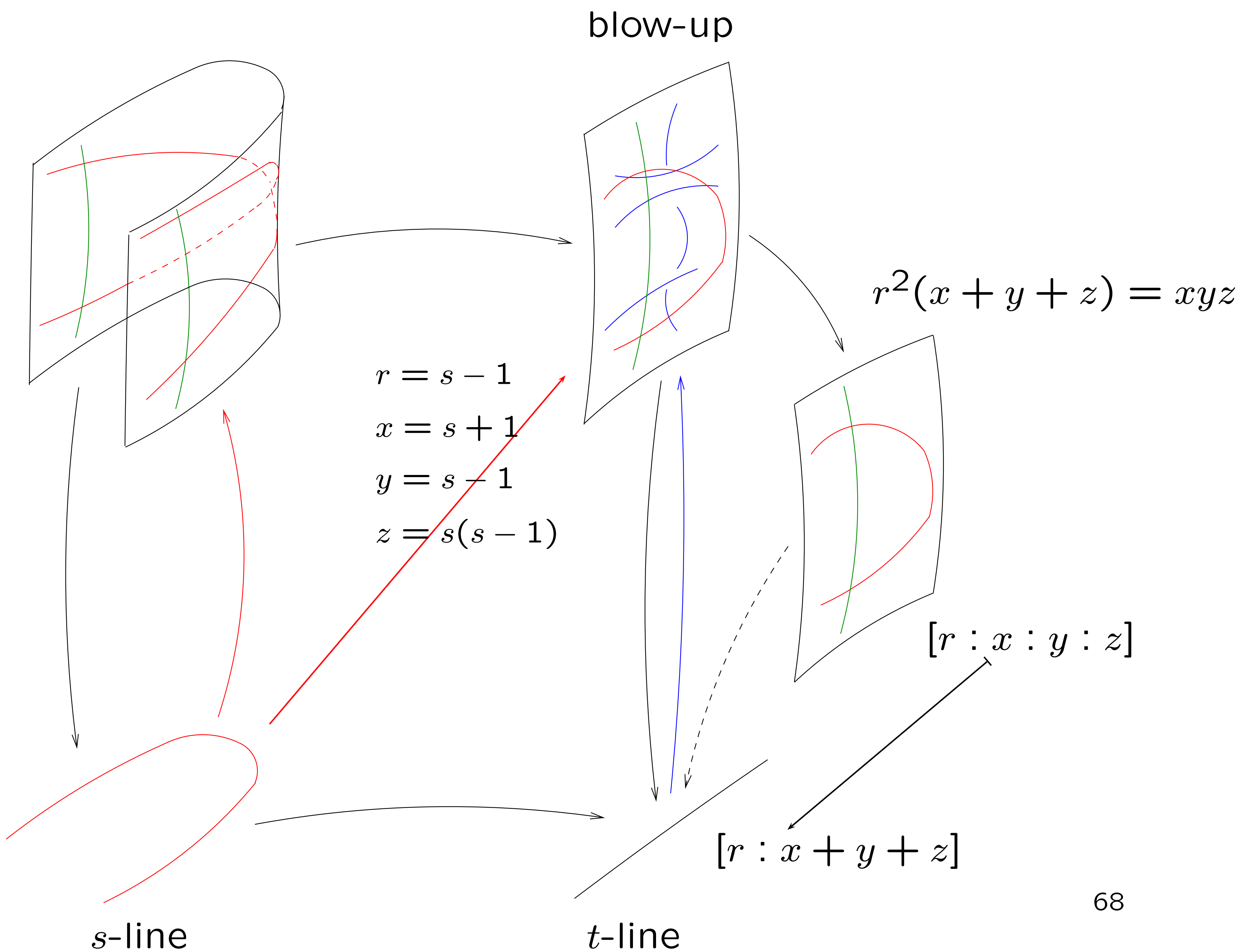


$r^2(x+y+z) = xyz$
$[r : x : y : z]$
$[r : x+y+z]$
t-line

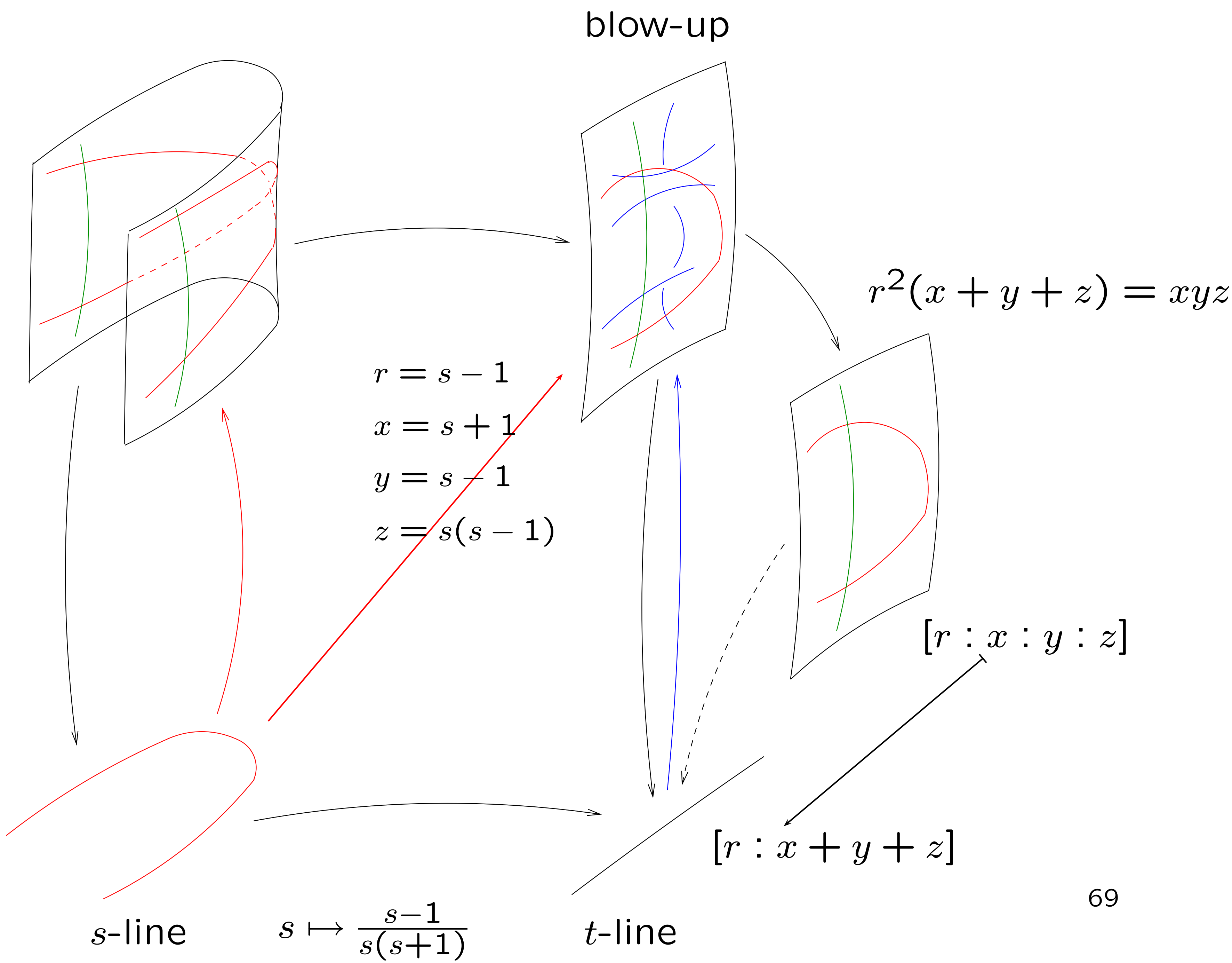

blow-up
$r^2(x+y+z) = xyz$
$[r : x : y : z]$
$[r : x+y+z]$
t-line



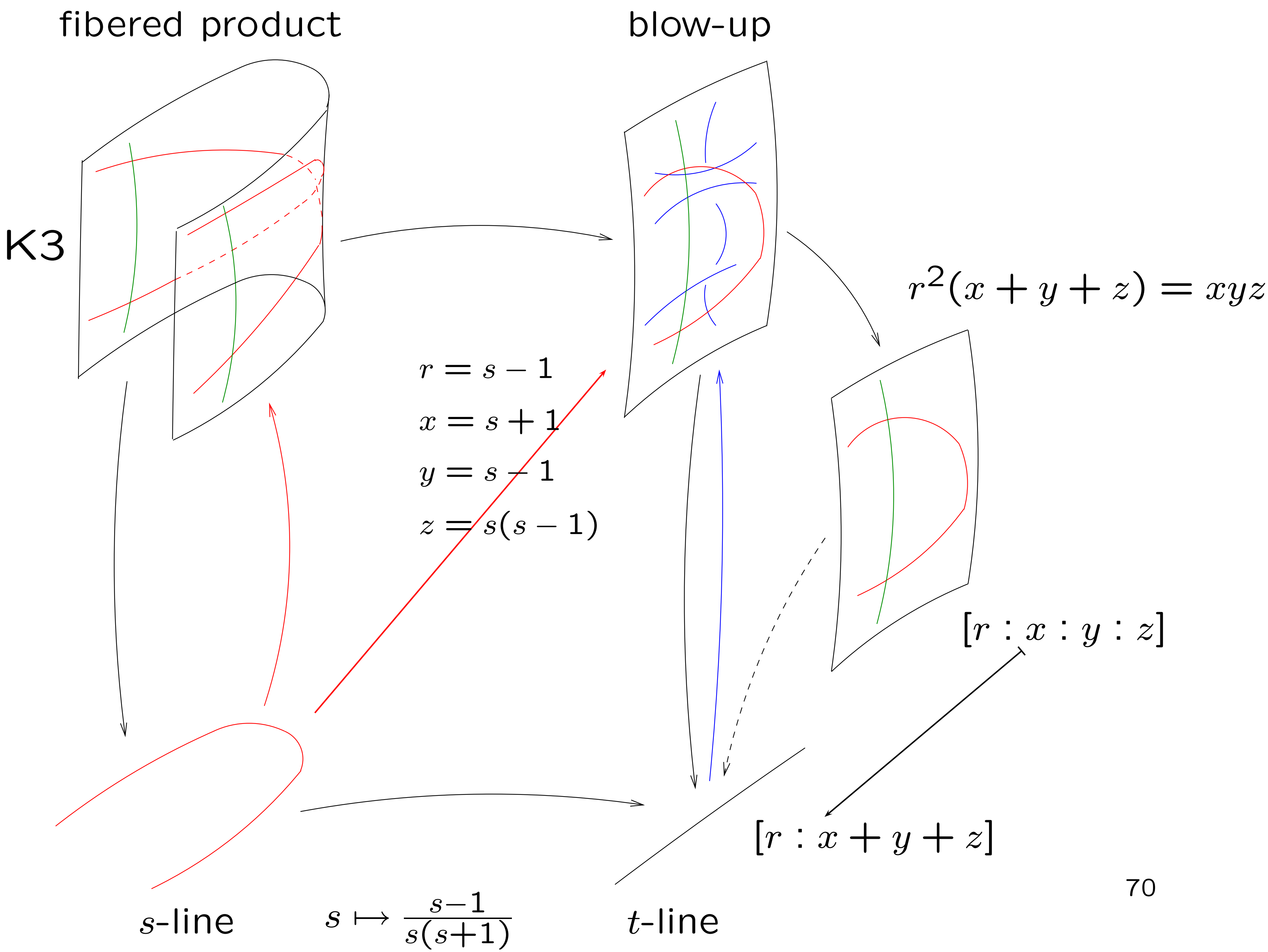
blow-up
$r^2(x+y+z) = xyz$
$[r : x : y : z]$
$[r : x+y+z]$
s-line
t-line



blow-up
$r^2(x+y+z) = xyz$
$r = s - 1$
$x = s + 1$
$y = s - 1$
$z = s(s-1)$
$[r : x : y : z]$
$[r : x+y+z]$
s-line
t-line



blow-up
$r^2(x+y+z) = xyz$
$r = s - 1$
$x = s + 1$
$y = s - 1$
$z = s(s-1)$
$[r : x : y : z]$
$[r : x + y + z]$
s-line
$s \mapsto \frac{s-1}{s(s+1)}$
t-line


fibered product
blow-up
K3
$r^2(x+y+z) = xyz$
$r = s - 1$
$x = s + 1$
$y = s - 1$
$z = s(s-1)$
$[r : x : y : z]$
$[r : x + y + z]$
s-line
$s \mapsto \frac{s-1}{s(s+1)}$
t-line

K3 surfaces are 2-dimensional Calabi-Yau manifolds.

Examples of K3 surfaces are smooth quartic surfaces in $\mathbb{P}^3$.

We showed that the K3 surface has infinitely many fibers with infinitely many rational points.

The set of rational points is dense on the K3 surface.

K3 surfaces are 2-dimensional Calabi-Yau manifolds.

Examples of K3 surfaces are smooth quartic surfaces in $\mathbb{P}^3$.

We showed that the K3 surface has infinitely many fibers with infinitely many rational points.

The set of rational points is dense on the K3 surface.

Open Problem:
Is there a K3 surface on which the set of rational points is neither empty, nor dense?

Swinnerton-Dyer suspected perhaps the quartic surface

$$x^4 + 2y^4 = z^4 + 4w^4$$

has only 2 rational points.

Swinnerton-Dyer suspected perhaps the quartic surface

$$x^4 + 2y^4 = z^4 + 4w^4$$

has only 2 rational points.

However, in 2004 Elsenhans and Jahnel found

$$1484801^4 + 2 \cdot 1203120^4 \ = \ 1169407^4 + 4 \cdot 1157520^4$$