An Elliptic K3 surface associated to Heron triangles

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October 7, 2005
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Goals:

(1) Solve a Diophantine problem

(2) Sketch an algebraic geometric point of view

(3) Open problems
A Heron triangle is a triangle with integral sides and integral area.

\[16A^2 = (a + b + c)(a + b - c)(a - b + c)(-a + b + c)\]

(Heron’s formula)
Examples of Heron triangles

Pythagorean triangles

\[ A = mn(m^2 - n^2) \]

\[ 2mn \]
Examples of Heron triangles

15 13 84
14
Examples of Heron triangles

\[
\begin{align*}
&15 & 12 & 13 \\
&54 & 30 & \\
&9 & 5 &
\end{align*}
\]
Examples of Heron triangles

Fact All Heron triangles can be obtained by glueing together Pythagorean triangles.
Examples of Heron triangles

\[ a = u(v^2 + w^2) \quad 0 < v < u \]
\[ b = v(u^2 + w^2) \quad 0 < w \]
\[ c = (u - v)(uv + w^2) \]
\[ A = uvw(u - v)(uv + w^2) \]

What more could we want than a parametrization?
Pairs of Heron triangles

25 17
28

20 29
21
Pairs of Heron triangles

- Triangle 1: 25, 17, 28
- Triangle 2: 20, 29, 210
Pairs of Heron triangles
Pairs of Heron triangles

Same area and same perimeter!
Infinitely many pairs (Aassila, Kramer & Luca):

\[ a_1 = t^{10} + 6t^8 + 15t^6 + 19t^4 + 11t^2 + 1 \]
\[ b_1 = t^{10} + 5t^8 + 10t^6 + 10t^4 + 6t^2 + 3 \]
\[ c_1 = t^8 + 5t^6 + 9t^4 + 7t^2 + 2 \]

\[ a_2 = t^{10} + 6t^8 + 15t^6 + 18t^4 + 9t^2 + 1 \]
\[ b_2 = t^{10} + 6t^8 + 14t^6 + 16t^4 + 9t^2 + 2 \]
\[ c_2 = t^6 + 4t^4 + 6t^2 + 3 \]

\[ p = 2t^{10} + 12t^8 + 30t^6 + 38t^4 + 24t^2 + 6 \]
\[ A = t(t^2 + 1)^4(t^2 + 2)(t^4 + 3t^3 + 3) \]
First Question:

Are there $n$-tuples of Heron triangles with the same area and the same perimeter for $n \geq 3$?
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Are there $n$-tuples of Heron triangles with the same area and the same perimeter for $n \geq 3$?

Answer:

Yes, in fact there are for any $n \geq 1$!
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\[ p = a + b + c = 6111518179503708972000 \]
\[ A = 1340792724147847711994993266314426038400000 \]
Main Question:
Are there parametrizations of $n$-tuples of Heron triangles with the same area and the same perimeter for $n \geq 3$?
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Are there parametrizations of \( n \)-tuples of Heron triangles with the same area and the same perimeter for \( n \geq 3 \)?

Answer:
Yes, such parametrizations exist for any \( n \).
New variables

c

b

a
New variables

\[ a \]

\[ b \]

\[ c \]
New variables

c

b

a
New variables

\[ a, b, c \]
New variables

\[ \text{diagram with points labeled } a, b, c, x, y, z, r \]
New variables

\[ s = \frac{1}{2}(a + b + c) \]
\[ s = x + y + z \]
New variables

\[
s = \frac{1}{2}(a + b + c)
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\[ A = rs \]
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\[ r^2(x + y + z) = xyz \]
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We want constant \( \frac{r}{x+y+z} \)
Every triangle gives $x, y, z, r > 0$ satisfying

$$r^2(x + y + z) = xyz.$$ 

Triangles with the same area and perimeter yield the same ratio

$$t = \frac{r}{x + y + z}.$$
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For fixed parameter $t$ we want solutions $x, y, z > 0$ to

$$t^2(x + y + z)^3 = xyz.$$
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$$t = \frac{r}{x + y + z}.$$  

For fixed parameter $t$ we want solutions $x, y, z > 0$ to

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We can find solutions for $t$ of the form

$$t = \frac{s - 1}{s(s + 1)}.$$  


Theorem 1. There exists a sequence \( \{(x_n, y_n, z_n)\}_{n \geq 1} \) of triples of elements in \( \mathbb{Q}(s) \) such that

1. for all \( n \geq 1 \) and all \( \sigma \in \mathbb{R} \) with \( \sigma > 1 \), there exists a triangle \( \Delta_n(\sigma) \) with sides \( y_n(\sigma) + z_n(\sigma) \), \( x_n(\sigma) + z_n(\sigma) \), and \( x_n(\sigma) + y_n(\sigma) \), and inradius \( (\sigma - 1)\sigma^{-1}(\sigma + 1)^{-1}(x_n(\sigma) + y_n(\sigma) + z_n(\sigma)) \), and

2. for all \( m, n \geq 1 \) and \( \sigma_0, \sigma_1 \in \mathbb{Q} \) with \( \sigma_0, \sigma_1 > 1 \), the rational triangles \( \Delta_m(\sigma_0) \) and \( \Delta_n(\sigma_1) \) are similar if and only if \( m = n \) and \( \sigma_0 = \sigma_1 \).
First triples in the sequence:

\[(x_1, y_1, z_1) = (1 + s, -1 + s, (-1 + s)s),\]
\[x_2 = (-1 + s)(1 + 6s - 2s^2 - 2s^3 + s^4)^3,\]
\[y_2 = (-1 + s)(-1 + 4s + 4s^2 - 4s^3 + s^4)^3,\]
\[z_2 = s(1 + s)(3 + 4s^2 - 4s^3 + s^4)^3,\]
\[x_3 = (-1 + s)(1 + 2s + 2s^2 - 2s^3 + s^4)^3\]
\[\quad ( -1 - 22s + 66s^2 + 14s^3 - 72s^4 + 30s^5 + 6s^6 - 6s^7 + s^8)^3,\]
\[y_3 = (1 + s)(-1 + 20s + 68s^2 - 84s^3 + 139s^4 + 32s^5 - 224s^6 +\]
\[\quad 64s^7 + 149s^8 - 148s^9 + 60s^{10} - 12s^{11} + s^{12})^3,\]
\[z_3 = (-1 + s)s(5 + 10s + 126s^2 + 62s^3 - 225s^4 + 52s^5 + 28s^6 +\]
\[\quad 12s^7 + 27s^8 - 62s^9 + 38s^{10} - 10s^{11} + s^{12})^3,\]
\[x_4 = (1 + s)(-1 - 62s + 198s^2 + 1698s^3 + 7764s^4 - 8298s^5 - 10830s^6 + 43622s^7 - 15685s^8 -\]
\[\quad 45356s^9 - 1348s^{10} + 75284s^{11} - 13088s^{12} - 93076s^{13} + 85220s^{14} + 12s^{15} - 49467s^{16} +\]
\[\quad 40842s^{17} - 16034s^{18} + 2282s^{19} + 844s^{20} - 546s^{21} + 138s^{22} - 18s^{23} + s^{24})^3,\]
\[y_4 = (-1 + s)(-1 + 54s + 550s^2 - 10s^3 + 5092s^4 + 16674s^5 + 98s^6 - 51662s^7 + 22875s^8 +\]
\[\quad 41916s^9 - 63076s^{10} + 45628s^{11} + 13088s^{12} - 63644s^{13} + 38884s^{14} + 17668s^{15} -\]
\[\quad 31195s^{16} + 8302s^{17} + 8990s^{18} - 9554s^{19} + 4476s^{20} - 1254s^{21} + 218s^{22} - 22s^{23} + s^{24})^3,\]
\[z_4 = (-1 + s)s(-7 - 28s - 1168s^2 - 2588s^3 + 5170s^4 + 6940s^5 + 20176s^6 - 10628s^7 -\]
\[\quad 70305s^8 + 46664s^9 + 85440s^{10} - 107832s^{11} + 380s^{12} + 66840s^{13} - 46848s^{14} + 13656s^{15} -\]
\[\quad 1465s^{16} - 2796s^{17} + 5712s^{18} - 5228s^{19} + 2738s^{20} - 884s^{21} + 176s^{22} - 20s^{23} + s^{24})^3.\]
Sketch of a very unenlightening proof:

**Step 1. Show there is one solution to**

\[
\frac{(s - 1)^2}{s^2(s + 1)^2}(x + y + z)^3 = xyz.
\]
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A solution is \((x, y, z) = (s + 1, s - 1, s(s - 1))\) (positive for \(s > 1\)).
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A solution is \((x, y, z) = (s + 1, s - 1, s(s - 1))\) (positive for \(s > 1\)).

**Step 2. Show that if there is a solution, then there are infinitely many solutions.**

If \((x, y, z)\) is a solution, then so is \((x', y', z')\) with …
\[ x' = -3(s + 1)(2s^2 - s + 1)(s - 1)^3z^3 - 2(s + 1)(4s^3 - 10s^2 + 7s - 3)(s - 1)^2xz^2 \\
+ 3(2s^6 - s^4 + 3s^3 + 5s^2 - 3s + 2)(s - 1)^2yz^2 - 3(s + 1)(-1 + s)^5x^2z \\
+ 3s(s^4 + s^2 + 2)(s - 1)^3y^3 + s(s^4 - s^3 + s^2 - 3s + 6)(s - 1)^3x^2y \\
- (s - 1)(s^9 + s^8 - 6s^7 + 15s^6 - 6s^5 + 6s^4 - 14s^3 + 31s^2 - 15s + 3)y^2z \\
+ 3s(s^4 - s^3 + s^2 - s + 4)(s - 1)^3xy^2 + 3(s^2 + 2s + 2)(s - 1)^6xyz, \]

\[ y' = -3(s + 1)(s^4 - s^3 + 4s^2 - s + 1)(s - 1)^2z^3 + 3s(s^2 + s + 2)(s - 1)^4y^3 \\
+ (s^9 - s^8 + 4s^7 + 6s^6 - 3s^5 + 9s^4 + 46s^3 - 40s^2 + 16s - 6)yz^2 + 6s(s - 1)^4x^2y \\
- (s + 1)(s^4 - 4s^3 + 10s^2 - 6s + 3)(s - 1)^2x^2z - 6(5s^2 - 2s + 1)(s - 1)^2xyz \\
- 3(s^6 + 4s^4 + 3s^3 + 10s^2 - 3s + 1)(s - 1)^2y^2z - s(s^3 - 2s^2 - 3s - 12)(s - 1)^4xy^2 \\
- 3(s + 1)(s^4 - 3s^3 + 7s^2 - 3s + 2)(s - 1)^2xz^2, \]

\[ z' = -s(-1 + s)(s + 1)^4z^3 + 3s^2(-1 + s)^2(s + 1)^3yz^2 + s^4(s + 1)(-1 + s)^4y^3 \\
- 3s^3(s + 1)^2(-1 + s)^3y^2z. \]
Problem 1: Even if $x, y, z > 0$, we may not have $x', y', z' > 0$. 
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**Problem 3:** Are the triangles nonsimilar?
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**Solution:** Do the same operation twice (ouch) …

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**Problem 3:** Are the triangles nonsimilar?

**Solution:** We need an interpretation of these formulas.
The change of variables

\[
\begin{align*}
x &= -s(s + 1)p + q \\
y &= -s(s + 1)p - q \\
z &= 8s(s + 1)(s - 1)^2
\end{align*}
\]

shows that our curve in \( \mathbb{P}^2 \) with parameter \( s \) given by

\[
(s - 1)^2(x + y + z)^3 = s^2(s + 1)^2xyz
\]

is isomorphic to the curve given by

\[
q^2 = (p - 4(s - 1)^2)^3 + s^2(s + 1)^2p^2
\]

together with the point at infinity.
Interlude on elliptic curves

**Definition:**
An **elliptic curve** is a curve given by an equation of the form

\[ y^2 = f(x) \]

with \( f \) a separable polynomial of degree 3, together with the point at infinity.

Elliptic curves are

- 1-dimensional lie-groups,

- Calabi-Yau 1-manifolds,

- 1-dimensional abelian varieties.
\( y^2 = x^3 + 5x^2 - 6x \)
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**Graphical Representation**

- **Point P**: $(-3, -6)$
- **Point Q**: $(2, 4)$
- **Point T**: $(0, 0)$
- **Point $P + T$**: $(2, -4)$

The graph shows the curve $y^2 = x^3 + 5x^2 - 6x$ with specific points marked on it.
\[ y^2 = x^3 + 5x^2 - 6x \]

\[
\left( -\frac{96}{25}, \frac{792}{125} \right) = 2P + T
\]

\[
(0, 0) = T
\]

\[
P = (-3, -6)
\]

\[
Q = (2, 4)
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Point ON egg + point ON egg = point ON egg

Point ON egg + point OFF egg = point ON egg

Point ON egg + point OFF egg = point OFF egg

53
Back to our curve

\[(s - 1)^2(x + y + z)^3 = s^2(s + 1)^2xyz\]

and the isomorphism

\[
x = -s(s + 1)p + q \\
y = -s(s + 1)p - q \\
z = 8s(s + 1)(s - 1)^2
\]

to the curve given by

\[q^2 = (p - 4(s - 1)^2)^3 + s^2(s + 1)^2p^2.\]
Back to our curve

$$(s - 1)^2(x + y + z)^3 = s^2(s + 1)^2xyz$$

and the isomorphism

$$x = -s(s + 1)p + q$$
$$y = -s(s + 1)p - q$$
$$z = 8s(s + 1)(s - 1)^2$$

to the curve given by

$$q^2 = (p - 4(s - 1)^2)^3 + s^2(s + 1)^2p^2.$$ 

For $s > 1$ the inequalities $x, y, z > 0$ are equivalent with

$p < 0, \quad \text{and} \quad s^2(s + 1)^2p^2 > q^2 = (p - 4(s - 1)^2)^3 + s^2(s + 1)^2p^2,$

so just to $p < 0$. This turns out to be equivalent to

“lying on the egg-part of the elliptic curve.”
Our simple solution \((x, y, z) = (s + 1, s - 1, s(s - 1))\) corresponds to the point \(R = (8 - 8s, 8s^2 - 8)\) on the egg-part of the elliptic curve for \(s > 1\).

The point \(P = (4(s - 1)^2, 4s(s + 1)(s - 1)^2)\) has order 3.
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The point \(P = (4(s - 1)^2, 4s(s + 1)(s - 1)^2)\) has order 3.

1. \(R\) has infinite order for rational \(s\) (Mazur’s Theorem),
2. All odd multiples of \(R\) lie on the egg-part,
3. \(mR\) and \(nR\) give similar triangles iff \(mR = \pm nR + kP\) for some \(k\),
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(2) All odd multiples of \(R\) lie on the egg-part,
(3) \(mR\) and \(nR\) give similar triangles iff \(mR = \pm nR + kP\) for some \(k\),

We conclude that the positive odd multiples of \(R\) give our wanted parametrizations.
Our simple solution \((x, y, z) = (s + 1, s - 1, s(s - 1))\) corresponds to the point \(R = (8 - 8s, 8s^2 - 8)\) on the egg-part of the elliptic curve for \(s > 1\).

The point \(P = (4(s - 1)^2, 4s(s + 1)(s - 1)^2)\) has order 3.

(1) \(R\) has infinite order for rational \(s\) (Mazur’s Theorem),
(2) All odd multiples of \(R\) lie on the egg-part,
(3) \(mR\) and \(nR\) give similar triangles iff \(mR = \pm nR + kP\) for some \(k\),

We conclude that the positive odd multiples of \(R\) give our wanted parametrizations.

(4) big formula \((x', y', z')\) corresponds to \(Q \mapsto Q + R\).
Unsatisfying:
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- Where did the substitution

\[ t = \frac{s - 1}{s(s + 1)} \]

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  come from?

- Where are the surfaces from the title of this talk?
$r^2(x + y + z) = xyz$
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\[ [r : x : y : z] \]

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\[
\begin{align*}
  r &= s - 1 \\
  x &= s + 1 \\
  y &= s - 1 \\
  z &= s(s - 1)
\end{align*}
\]
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\begin{align*}
    r &= s - 1 \\
    x &= s + 1 \\
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\end{align*}
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\[ s \mapsto \frac{s-1}{s(s+1)} \]
$r^2(x + y + z) = xyz$

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$y = s - 1$

$z = s(s - 1)$

$[r : x : y : z]$

$[r : x + y + z]$
K3 surfaces are 2-dimensional Calabi-Yau manifolds.

Examples of K3 surfaces are smooth quartic surfaces in $\mathbb{P}^3$.

We showed that the K3 surface has infinitely many fibers with infinitely many rational points.

The set of rational points is dense on the K3 surface.
K3 surfaces are 2-dimensional Calabi-Yau manifolds.

Examples of K3 surfaces are smooth quartic surfaces in $\mathbb{P}^3$.

We showed that the K3 surface has infinitely many fibers with infinitely many rational points.

The set of rational points is dense on the K3 surface.

**Open Problem:**
Is there a K3 surface on which the set of rational points is neither empty, nor dense?
Swinnerton-Dyer suspected perhaps the quartic surface

\[ x^4 + 2y^4 = z^4 + 4w^4 \]

has only 2 rational points.
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\[ x^4 + 2y^4 = z^4 + 4w^4 \]
has only 2 rational points.

However, in 2004 Elsenhans and Jahnel found
\[ 1484801^4 + 2 \cdot 1203120^4 = 1169407^4 + 4 \cdot 1157520^4 \]