Explicit computations on the Manin conjectures

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Describing the set of rational points on a variety

curve $C/\mathbb{Q}$ of genus g
$C(\mathbb{Q}) = \emptyset$ $C(\mathbb{Q}) = \{P_1, \ldots, P_n\}$ $C(\mathbb{Q})$ is dense in C: • fin. gen. group ($g = 1$) • $\exists$ a parametrization ($g = 0$)

satisfying answers

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curve $C/\mathbb{Q}$ of genus g	X of dimension $d > 1$
$C(\mathbb{Q}) = \emptyset$	dim(Zariski closure) $<$ d
$C(\mathbb{Q}) = \{P_1, \ldots, P_n\}$	$X(\mathbb{Q})$ is dense in X:
$C(\mathbb{Q})$ is dense in C:	• fin. gen. grp. (abelian var.)
• fin. gen. group ($g = 1$)	• $\exists$ parametrization (rat. var.)
• $\exists$ a parametrization ($g = 0$)	• ???

satisfying answers — not so much

Measuring the number of points

Let $X \subset \mathbb{P}^n/\mathbb{Q}$ be smooth, geometrically integral, projective.

Let the **height** $H \colon \mathbb{P}^n(\mathbb{Q}) \to \mathbb{R}_{>0}$ be defined by

$$H(x) = \max_i(|x_i|) \qquad \text{if } \begin{cases} x = [x_0 : x_1 : \ldots : x_n] \\ x_i \in \mathbb{Z} \\ \gcd(x_0, \ldots, x_n) = 1 \end{cases}$$

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For any open $U \subset X$ we set

$$N_U(B) = \#\{x \in U(\mathbb{Q}) \ : \ H(x) \leq B\}.$$

We want to understand the asymptotic behavior of N_U.

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Examples

(1) $N_{\mathbb{P}^n}(B) \approx \frac{1}{2}(2B+1)^{n+1}\prod_{p<B}\left(1 - \frac{1}{p^{n+1}}\right) \approx \frac{2^n B^{n+1}}{\zeta(n+1)}$

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Fact: After the Segre embedding into $\mathbb{P}^{rs+r+s}$, the height on the product of $X_1 \subset \mathbb{P}^r$ and $X_2 \subset \mathbb{P}^s$ is equal to the product of their heights.

(2) $X = \mathbb{P}^1 \times \mathbb{P}^1$, i.e., a quadric in $\mathbb{P}^3$

$$N_{\mathbb{P}^1 \times \mathbb{P}^1}(B) \ \approx \ CB^2 \log(B)$$

Sometimes a large contribution comes from a small set

(3) Let $X \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be given by $x_1 x_2 x_3 = y_1 y_2 y_3$.
Let E_{ij} be the line $x_i = y_j = 0$ for $i \neq j$, and $U = X - \bigcup E_{ij}$.

$$N_U(B) \approx \frac{1}{6}\left(\prod_p \left(1 - \frac{1}{p}\right)^4 \left(1 + \frac{4}{p} + \frac{1}{p^2}\right)\right) B(\log B)^3$$

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In all cases there are C, a, b, U such that

$$N_U(B) \;\approx\; CB^a(\log B)^b$$

Question: how are C, a, b related to the geometry of X?

K_X is canonical divisor, H is a hyperplane section, $\rho = \mathrm{rk}\,\mathrm{NS}(X)$

X	$-K_X$	ρ	$\exists U, C : N_U(B) \approx$
$\mathbb{P}^n$	$(n+1)H$	1	CB^{n+1}
$\mathbb{P}^1 \times \mathbb{P}^1$ quadric in $\mathbb{P}^3$	$2H$	2	$CB^2 \log B$
$x_1x_2x_3 = y_1y_2y_3$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ del Pezzo of deg 6 in $\mathbb{P}^6 \subset \mathbb{P}^7$	H	4	$CB(\log B)^3$

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X	$aH, a > 0$	$b+1$	$CB^a(\log B)^b$ **?**

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X	$aH, a > 0$	$b+1$	$CB^a(\log B)^b$?

Problem: may need a finite field extension to avoid obstructions

Conjecture 1 (Batyrev, Manin). Let X be a smooth, geometrically integral, projective variety over a number field k, and let H be a hyperplane section. Assume that the canonical sheaf K_X satisfies $-K_X = aH$ for some $a > 0$. Then there exists a finite field extension l, a constant C, and an open subset $U \subset X$, such that with $b = \operatorname{rk} \mathrm{NS}(X_l) - 1$ we have

$$N_{U_l}(B) \ \approx \ CB^a(\log B)^b.$$

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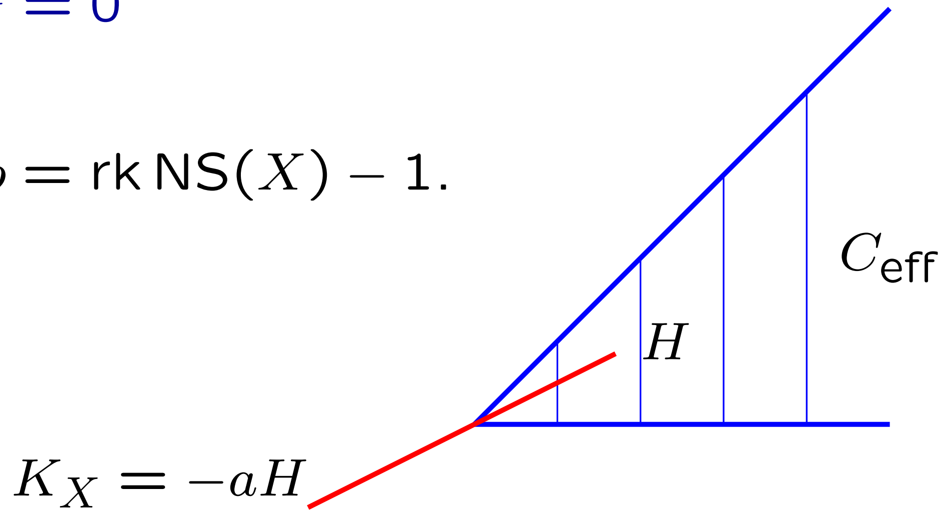
Conjecture 2 (generalization). The same, except that K_X is only assumed not to be effective. If C_{eff} denotes the closed cone inside $\mathrm{NS}(X_l)_{\mathbb{R}}$ generated by effective divisors, then a and b are given by

$$\begin{aligned} a &= \inf\{\gamma \in \mathbb{R} : \gamma H + K_X \in C_{\mathrm{eff}}\} \\ b+1 &= \text{the codimension of the minimal face of} \\ &\qquad \partial C_{\mathrm{eff}} \text{ containing } aH + K_X. \end{aligned}$$

Limiting case, $K_X = 0$

We get $a = 0$ and $b = \mathrm{rk}\,\mathrm{NS}(X) - 1$.



We will only consider K3 surfaces.

Then the asymptotics are probably not true in general, as such a surface may contain an elliptic fibration with infinitely many fibers contributing too many points.

$\underbrace{\text{K3 surfaces}}_{a=0}$ X with $\underbrace{\operatorname{rk}\mathrm{NS}(X)=1}_{b=0}$ do not admit such a fibration.

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Let $X \subset \mathbb{P}^3$ be given by

$$x^4 + 2y^4 = z^4 + 4w^4.$$

Then $\mathrm{rk}\, NS(X) = 1$ (over $\mathbb{Q}$).

Question (Swinnerton-Dyer, 2002):

Does X have more than 2 rational points?

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Answer (Elsenhans, Jahnel, 2004):

$$1484801^4 + 2 \cdot 1203120^4 \ = \ 1169407^4 + 4 \cdot 1157520^4$$

Theorem (vL, 2004)
The K3 surface X in $\mathbb{P}^3$ given by

$$w(x^3+y^3+z^3+x^2z+xw^2) = 3x^2y^2-4x^2yz+x^2z^2+xy^2z+xyz^2-y^2z^2$$

is smooth and satisfies $\operatorname{rk}\mathrm{NS}(X_{\overline{\mathbb{Q}}}) = 1$.

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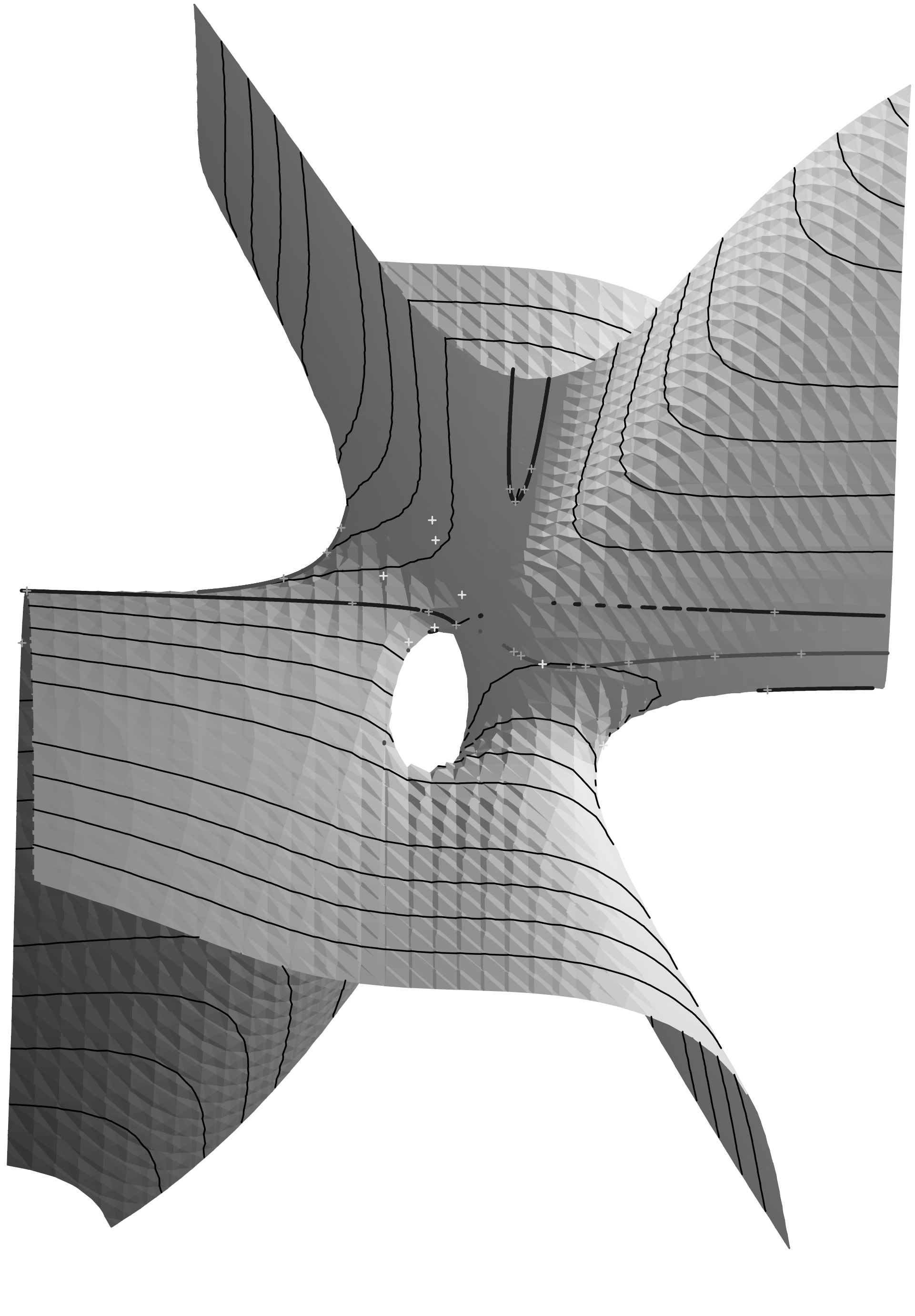
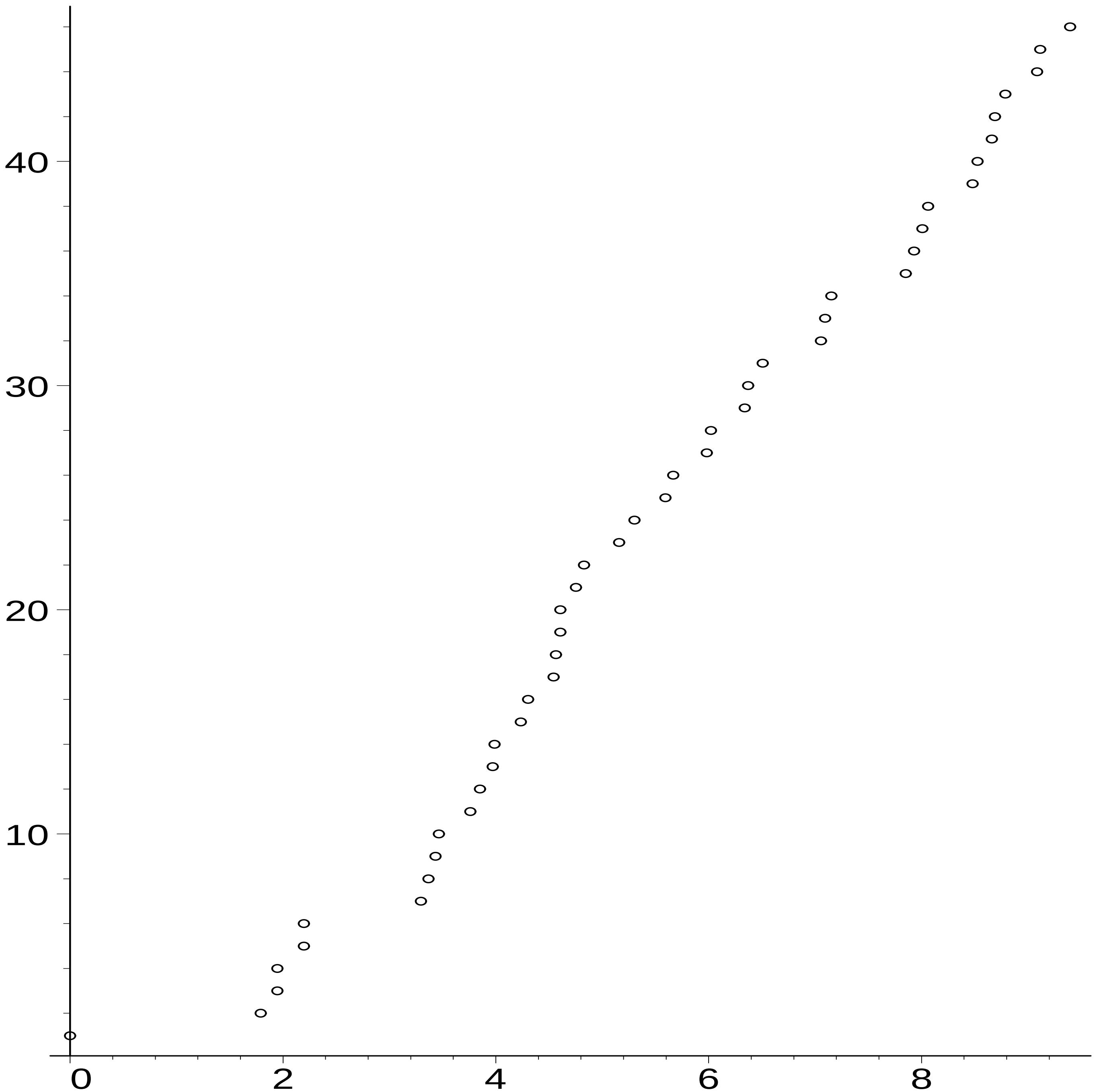
Sketch of proof

- $\operatorname{NS}(X_{\overline{\mathbb{Q}}}) \hookrightarrow \operatorname{NS}(X_{\overline{\mathbb{F}_p}})$ for primes p of good reduction.
- $\operatorname{rk}\operatorname{NS}(X_{\overline{\mathbb{F}_p}}) = 2$ for $p = 2, 3$.
- $\operatorname{NS}(X_{\overline{\mathbb{F}_2}})_{\mathbb{Q}} \ncong \operatorname{NS}(X_{\overline{\mathbb{F}_3}})_{\mathbb{Q}}$ as inner product spaces.

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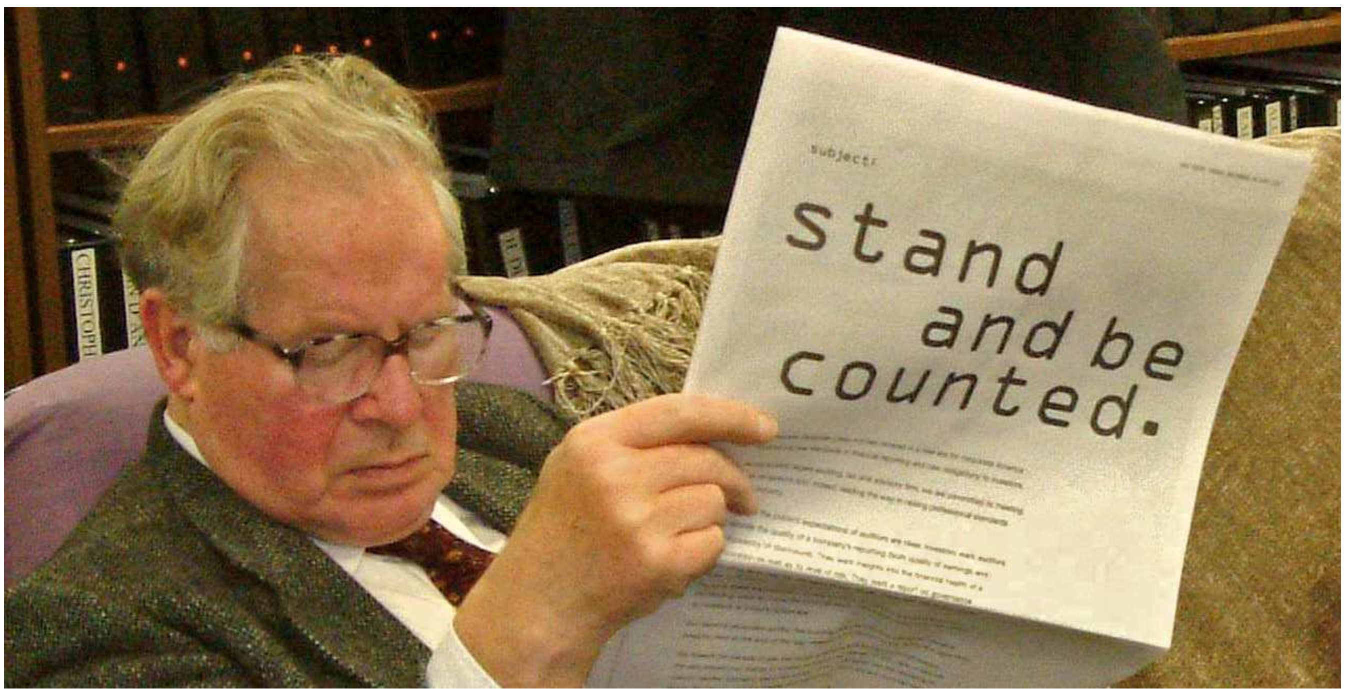

Picture taken by William Stein

$$\int B^{a-1}\,dB = \begin{cases} CB^a & \text{if } a \neq 0 \\ \log(B) & \text{if } a = 0 \end{cases}$$

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Questure: Let X be a K3 surface over a number field k with $\operatorname{rk} \mathrm{NS}(X_{\overline{k}}) = 1$. Is there a finite field extension l, a constant C, and an open subset $U \subset X$, such that U contains no curve of genus 1 over l and

$$N_{U_l}(B) \;\approx\; C \log B?$$