Concurrent lines on Del Pezzo surfaces of degree one

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Cubic surfaces

Let $X \subset \mathbb{P}^3$ be a smooth cubic surface over a field $k = \overline{k}$.

**Theorem** (Cayley-Salmon).
The surface $X$ contains exactly 27 lines.
Each point lies on at most 3 lines.
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**Proof of second part.**
The lines through a point $P$ lie in the tangent plane $H$,
so in the cubic curve $X \cap H$.
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**Proof of second part.**
The lines through a point $P$ lie in the tangent plane $H$, so in the cubic curve $X \cap H$.

**Definition.**
An Eckardt point is a point that lies on three lines.
Cubic surfaces

Example.
The Fermat surface \( w^3 + x^3 + y^3 + z^3 = 0 \) contains the 9 lines

\[
  w^3 + x^3 = y^3 + z^3 = 0.
\]

Three of these go through \([1 : -1 : 0 : 0]\).
In total: 27 lines and 18 Eckardt points.
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\[ w^3 + x^3 = y^3 + z^3 = 0. \]

Three of these go through $[1 : -1 : 0 : 0]$.
In total: 27 lines and 18 Eckardt points.

Fact. There are 45 tritangent planes.

Fact (Hirschfeld ’67).
At most 45 Eckardt points (sharp in characteristic 2).
At most 18 Eckardt points in characteristic not equal to 2.
Cubic surfaces (another point of view)

**Fact.** The cubic surface $X \subset \mathbb{P}^3$ is isomorphic to the blow up of $\mathbb{P}^2$ in 6 points in general position: no 3 on a line, no 6 on a conic.
Cubic surfaces (another point of view)

**Fact.** The cubic surface $X \subset \mathbb{P}^3$ is isomorphic to the blow up of $\mathbb{P}^2$ in 6 points in general position: no 3 on a line, no 6 on a conic.

**Fact.** The canonical divisor $K_X$ is linearly equivalent with

$(-n - 1 + d)H = (-3 - 1 + 3)H = -H$ for any hyperplane section $H$.

**Fact** (Riemann-Roch and adjunction). The lines on $X$ correspond with the classes $E$ in $\text{Pic } X$ with $E^2 = -1$ and $H \cdot E = 1$. 
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**Fact.** $\text{Pic} X$ is the orthogonal direct sum $\mathbb{Z}L \oplus \bigoplus_{i=1}^6 \mathbb{Z}E_i$, where $\pi: X \to \mathbb{P}^2$ is the blow up, $L$ is the pull back of a line, and the $E_i$ are the exceptional curves. Moreover, $L^2 = 1$, $E_i^2 = -1$, $L \cdot E_i = 0$, $E_i \cdot E_j = 0$ ($i \neq j$).

**Corollary.** We can describe all lines in terms of $\text{Pic} X$. 
Corollary. The classes $E \in \text{Pic } X$ with $E^2 = -1$ and $-K_X \cdot E = 1$:

- $E_i$ for $1 \leq i \leq 6$,
- $L - E_i - E_j$ for $1 \leq i < j \leq 6$,
- $2L + E_i - \sum E_j$ for $1 \leq i \leq 6$,

corresponding to the 6 exceptional curves and strict transforms of the 15 lines through two points and the 6 conics through five of the six points.
Cubic surfaces (another point of view)

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**Corollary.** The incidence graph on the 27 lines is independent of $X$. It is the complement of the Schläfli graph.

**Schläfli graph.** Each line is connected to 10 other lines, partitioned in five disjoint pairs of intersecting lines.
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**Corollary.** The incidence graph on the 27 lines is independent of $X$. It is the complement of the Schläfli graph.

**Schläfli graph.** Each line is connected to 10 other lines, partitioned in five disjoint pairs of intersecting lines.

**Corollary.** (Concurrent lines form a complete subgraph, so) each point on at most 3 lines and at most $\frac{27 \cdot 5}{3} = 45$ Eckardt points.
Del Pezzo surfaces

A del Pezzo surface over a field \( k \) is a geometrically integral, smooth, projective surface \( S \) over \( k \) for which there exists an embedding \( i: S \hookrightarrow \mathbb{P}^n \) and a positive integer \( a \) such that the multiple \(-aK_S\) of a canonical divisor \( K_S \) is linearly equivalent to a hyperplane section. Its degree is \( \text{deg } S = K_S^2 \).

Examples.

- A smooth cubic surface in \( \mathbb{P}^3 \), with \( a = 1 \) and degree 3.
- A double cover of \( \mathbb{P}^2 \) ramified over a smooth quartic curve, with \( a = 2 \) (w.r.t. an embedding in \( \mathbb{P}^6 \)) and degree 2.

Fact.

Over an algebraically closed field, every del Pezzo surface is isomorphic to

- \( \mathbb{P}^1 \times \mathbb{P}^1 \), with degree 8, or
- \( \mathbb{P}^2 \) blown up at \( r \leq 8 \) points in general position (!), with degree \( 9 - r \) (cubic: \( r = 6 \)).
Del Pezzo surfaces

General position:

- no 3 points on a line,
- no 6 points on a conic,
- no 8 points on a singular cubic with singularity at one of 8.
Del Pezzo surfaces

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- no 8 points on a singular cubic with singularity at one of 8.

The $(-1)$-curves on the blow-up:
- the exceptional curves, and strict transforms of...
- lines through two (of the) points,
- conics through five points,
- cubics through seven points, singular at one (of them),
- quartics through eight points, singular at three,
- quintics through eight points, singular at six,
- sextics through eight singular points, triple point at one.

\[\begin{array}{c|cccccccc}
  r & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
  \# & 0 & 1 & 3 & 6 & 10 & 16 & 27 & 56 & 240 \\
\end{array}\]
Del Pezzo surfaces of degree two

These are double covers of $\mathbb{P}^2$, ramified over smooth quartic curve. The 28 bitangents pull back to 56 “lines”, that is, $(−1)$-curves. Each line $e$ intersects its partner $e'$ with multiplicity 2. Each other line intersects exactly one of $e$ and $e'$ with multiplicity 1.
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**Corollary.** Each line intersects 27 lines with multiplicity 1.

**Fact.** The subgraph on these 27 lines is the graph for cubics!

**Reason.** True for 27 lines intersected with multiplicity 0 and there is an automorphism that sends each line to its partner.
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**Corollary.** Each point of a del Pezzo surface $X$ of degree 2 lies on at most 4 lines (generalised Eckardt point: not on branch curve). There are at most $56 \cdot \frac{27}{4-1} \cdot \frac{1}{4} = 126$ generalised Eckardt points. This upper bound is sharp: $w^2 = x^4 + y^4 + z^4$ over $\mathbb{F}_9$.

**Question.** What about characteristics other than 3?
Del Pezzo surfaces: automorphism groups (Manin)

Let $X$ be the blow-up of $\mathbb{P}^2$ in $6 \leq r \leq 8$ points in general position.

Let $\mathcal{E}$ be the set of classes corresponding to lines. Then $\mathcal{E}$ lies in the hyperplane in the lattice $\text{Pic } X$ given by $-K_X \cdot e = 1$. 
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\[
\text{Aut Pic} X \supset \{ \sigma : \sigma K_X = K_X \} \xrightarrow{\cong} \text{Aut } K_X^\perp \xrightarrow{\cong} W(E_r)
\]

\[
\downarrow \cong \\
\text{Sym}(\mathcal{E}) \supset \{ \sigma : \sigma(e) \cdot \sigma(e') = e \cdot e' \text{ for all } e, e' \in \mathcal{E} \} =: G_r
\]

where $E_6, E_7, E_8$ are the classical root lattices (of $x$ with $x^2 = -2$).

\[
\# G_6 = 2^7 \cdot 3^4 \cdot 5 = 51840,
\]
\[
\# G_7 = 2^{10} \cdot 3^4 \cdot 5 \cdot 7 = 2903040,
\]
\[
\# G_8 = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7 = 696729600.
\]

**Fact.** The group $G_r$ acts transitively on $\mathcal{E}$.
Del Pezzo surfaces of degree one

Every del Pezzo surface $X$ of degree one is the double cover of a cone $C \subset \mathbb{P}^3$, ramified over a curve $B$ of degree 6: the intersection of $C$ with a surface of degree 3.
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**Fact.** There are 120 planes that are tritangent to $B$ and do not contain the vertex of the cone. They pull back to 240 lines on $X$. Each intersects its partner with multiplicity 3.
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**Fact.** If two partnered lines go through a point $P$, then $P$ lies on the ramification curve.

**Fact.** If $P$ lies on the ramification curve, then the partner of any line through $P$ also goes through $P$.

**Fact.** If $e$ and $e'$ are partners, then $e \cdot f = 2 - e' \cdot f$ for all lines $f$. ($e + e' \sim -2K_X$)
Del Pezzo surfaces of degree one

\[ L - e_1 - e_2 \]
Del Pezzo surfaces of degree one

**Fact** (Manin). The group $G = G_8$ acts transitively on

$$U = \{ (e_1, e_2, \ldots, e_8) \in E^8 : e_i \cdot e_j = 0 \text{ for } i \neq j \}.$$  

**Fact.** For every $u = (e_1, \ldots, e_8) \in U$, we can blow down $e_1, \ldots, e_8$ and there is a unique $\ell \in \text{Pic } X$ such that $-K_X = 3\ell - \sum_i e_i$. Since $G$ acts faithfully on $\text{Pic } X$, it acts freely on $U$, so $|U| = |G|$. 
Del Pezzo surfaces of degree one

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This action allows us to find the maximal complete subgraphs inside the graph on all 240 lines.
Lemma. The group $G$ acts transitively on the sets

\[ V_1 = \{ (e_1, e_2) : e_1 \cdot e_2 = 0 \} \]
\[ V_2 = \{ (e_0, e_1, e_2) : e_0 \cdot e_1 = e_0 \cdot e_2 = 1 \text{ and } e_1 \cdot e_2 = 0 \} \]
\[ V_3 = \{ (e_0, e_1) : e_0 \cdot e_1 = 1 \} . \]
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Proof. For $V_1$: the stabiliser $G_{e_1}$ is isomorphic to $G_7$ and the subgraph on $\{ e : e \cdot e_1 = 0 \}$ corresponds with $r = 7$. We have $\# V_1 = 240 \cdot 56$. 

Proof. For $V_2$: the stabiliser $G_{e_1}$ is isomorphic to $G_7$ and the subgraph on $\{ e : e \cdot e_1 = 0 \}$ corresponds with $r = 7$. We have $\# V_2 = 240 \cdot 56 \cdot 72$. For one specific $v = (e_0, e_1, e_2) \in V_2$, the set $W_v = \{ e : e \cdot e_0 = e \cdot e_1 = e \cdot e_2 = 0 \}$ has 6 elements, and the stabiliser $G_v$ injects into $\text{Sym}(W_v) \cong S_6$, so $\# G_v \leq 720$. Hence, the orbit has size $\# G_v = \# G \cdot \# G_v \geq \# G_7 \cdot 240 \cdot 56 \cdot 72$, so we have equality, so the action is transitive.

The $G$-action on the image of the projection $\rho : V_2 \rightarrow V_3$ is transitive. One fiber has size 32, so all non-empty fibers do. Hence, $\# \text{im} \rho = \# V_2 = 240 \cdot 56 \cdot 72 = 240 \cdot 126 = \# V_3$, so $\rho$ is surjective and $G$ acts transitively on $V_3$.

Corollary. For any $v = (e_0, e_1, e_2) \in V_2$, there is a blow-down $X \rightarrow \mathbb{P}^2$ such that $e_1, e_2$ are exceptional curves above two of the eight points blown up, and $e_0$ is the strict transform of the line in $\mathbb{P}^2$ through these two points.

Proof. Let $E_1, \ldots, E_8$ be the exceptional curves on $X$ above the eight points $P_1, \ldots, P_8 \in \mathbb{P}^2$ that were blown up to define $X$. Let $E_0$ be the strict transform of the line through $P_1$ and $P_2$. Let $g \in G \subset \text{Sym}(E)$ be an element that sends $(E_0, E_1, E_2)$ to $(e_0, e_1, e_2)$. Then $g$ sends $E_1, E_2, \ldots, E_8$ to elements $e_1, e_2, \ldots, e_8$ that we can blow down.
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One fiber of the map $V_2 \to V_1$ has size 72, so all fibers do, so $\#V_2 = 240 \cdot 56 \cdot 72$. For one specific $v = (e_0, e_1, e_2) \in V_2$, the set

\[ W_v = \{ e : e \cdot e_0 = e \cdot e_1 = e \cdot e_2 = 0 \} \]

has 6 elements, and the stabiliser $G_v$ injects into $\text{Sym}(W_v) \cong S_6$, so $\#G_v \leq 720$. Hence, the orbit has size

\[ \#G_v = \frac{\#G}{\#G_v} \geq \frac{\#G}{720} = \#V_2, \]

so we have equality, so the action is transitive.
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$$V_3 = \{ (e_0, e_1) : e_0 \cdot e_1 = 1 \}.$$

**Proof.**
The $G$-action on the image of the projection $\rho : V_2 \to V_3$ is transitive. One fiber has size 32, so all non-empty fibers do. Hence,

$$\# \text{im } \rho = \frac{\# V_2}{32} = \frac{240 \cdot 56 \cdot 72}{32} = 240 \cdot 126 = \# V_3,$$

so $\rho$ is surjective and $G$ acts transitively on $V_3$. 
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Corollary. For any $v = (e_0, e_1, e_2) \in V_2$, there is a blow-down $X \to \mathbb{P}^2$ such that $e_1, e_2$ are exceptional curves above two of the eight points blown up, and $e_0$ is the strict transform of the line in $\mathbb{P}^2$ through these two points.
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Proof. Let $E_1, \ldots, E_8$ be the exceptional curves on $X$ above the eight points $P_1, \ldots, P_8 \in \mathbb{P}^2$ that were blown up to define $X$. Let $E_0$ be the strict transform of the line through $P_1$ and $P_2$. Let $g \in G \subset \text{Sym}(\mathcal{E})$ be an element that sends

\[(E_0, E_1, E_2) \to (e_0, e_1, e_2).\]

Then $g$ sends $E_1, E_2, \ldots, E_8$ to elements $e_1, e_2, \ldots, e_8$ that we can blow down.
Del Pezzo surfaces of degree one (example over $\mathbb{F}_{32}$)

The maximal size of a complete subgraph is 16. There is an example with 16 concurrent lines!
Del Pezzo surfaces of degree one (example over $\mathbb{F}_{32}$)

The maximal size of a complete subgraph is 16.
There is an example with 16 concurrent lines!
Set $F = \mathbb{F}_2[\alpha] = \mathbb{F}_2[x]/(x^5 + x^2 + 1)$ and $P = [0 : 0 : 1] \in \mathbb{P}^2_F$ and:

\[ Q_1 = (0 : 1 : 1), \quad Q_5 = (1 : 1 : 1), \]
\[ Q_2 = (0 : 1 : \alpha^{19}), \quad Q_6 = (\alpha^4 : \alpha^4 : 1), \]
\[ Q_3 = (1 : 0 : 1), \quad Q_7 = (\alpha^{24} : \alpha^{25} : 1), \]
\[ Q_4 = (1 : 0 : \alpha^5), \quad Q_8 = (\alpha^{25} : \alpha^{26} : 1). \]

Then these curves go through $P$ (with $1 \leq i \leq 4$):

- the four lines through $Q_{2i-1}$ and $Q_{2i}$,
- the four cubics through all $Q_j$ with $j \neq 2i - 1$, singular at $Q_{2i}$,
- the four cubics through all $Q_j$ with $j \neq 2i$, singular at $Q_{2i-1}$,
- the four quintics through all $Q_j$, singular when $j \neq 2i, 2i - 1$.

Only need to check for 9 as these form eight partnered pairs.
Question. What about characteristic 0?
Two cases: the point $P$ on and off the ramification curve.
Del Pezzo surfaces of degree one

**Question.** What about characteristic 0? Two cases: the point $P$ on and off the ramification curve.

**Theorem** (Winter, vL). In both cases the number of concurrent lines is at most 10.
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Two cases: the point $P$ on and off the ramification curve.

**Theorem** (Winter, vL).
The number of concurrent lines is at most 10.

**Facts** (arguments similar to before, to minimise computation).
- Any 6 partnered pairs forming a complete subgraph are contained in a maximal clique of size 16, and $G$ acts transitively on the sets of 6 such pairs.
- Any 11 lines without partners forming a complete subgraph is contained in a clique of size 12 without partners, and $G$ acts transitively on the sets of 11 such lines in any such clique.
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**Question.** What about characteristic 0? Two cases: the point $P$ on and off the ramification curve.

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**Facts** (arguments similar to before, to minimise computation).

- Any 6 partnered pairs forming a complete subgraph are contained in a maximal clique of size 16, and $G$ acts transitively on the sets of 6 such pairs.
- Any 11 lines without partners forming a complete subgraph is contained in a clique of size 12 without partners, and $G$ acts transitively on the sets of 11 such lines in any such clique.

**Corollary.** To show that no 6 such pairs (or 11 such lines) are concurrent, it suffices to pick any description in $\mathbb{P}^2$ of 6 such pairs (or of 11 such lines in a clique of 12 such lines).
Del Pezzo surfaces of degree one

**Proposition (case $P$ on the ramification curve)**

Assume that $\text{char } k \neq 2$.

Let $Q_1, \ldots, Q_8$ be eight points in $\mathbb{P}^2$ in general position.

Let $L_i$ be the line through $Q_{2i}$ and $Q_{2i-1}$ for $i \in \{1, 2, 3, 4\}$, and $C_{i,j}$ the unique cubic through $Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_8$ that is singular in $Q_j$.

Assume that the four lines $L_1$, $L_2$, $L_3$ and $L_4$ all intersect in one point $P$. Then the three cubics $C_{7,8}$, $C_{8,7}$, and $C_{6,5}$ do not all go through $P$.

Proof. Gröbner bases with a lot of manual help.

Corollary. No six partnered pairs through one point, so no point on ramification curve lies on more than $10$ lines.

Fact. In each characteristic there is an example of a del Pezzo surface $X$ with $10$ concurrent lines.
Del Pezzo surfaces of degree one

**Proposition (case P on the ramification curve)**

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**Proof.** Gröbner bases with a lot of manual help.

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**Fact.** In each characteristic there is an example of a del Pezzo surface $X$ with 10 concurrent lines.
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Assume that \( \text{char } k \neq 2 \).
Let \( Q_1, \ldots, Q_8 \) be eight points in \( \mathbb{P}^2 \) in general position.
Let \( L_i \) be the line through \( Q_{2i} \) and \( Q_{2i-1} \) for \( i \in \{1, 2, 3, 4\} \),
and \( C_{i,j} \) the unique cubic through \( Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_8 \)
that is singular in \( Q_j \).
Assume that the four lines \( L_1, L_2, L_3 \) and \( L_4 \) all intersect in one point \( P \). Then the three cubics \( C_{7,8}, C_{8,7}, \) and \( C_{6,5} \) do not all go through \( P \).

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Assume that $\text{char } k \neq 2$.
Let $Q_1, \ldots, Q_8$ be eight points in $\mathbb{P}^2$ in general position.
Let $L_i$ be the line through $Q_{2i}$ and $Q_{2i-1}$ for $i \in \{1, 2, 3, 4\}$, and $C_{i,j}$ the unique cubic through $Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_8$ that is singular in $Q_j$.
Assume that the four lines $L_1$, $L_2$, $L_3$ and $L_4$ all intersect in one point $P$. Then the three cubics $C_{7,8}$, $C_{8,7}$, and $C_{6,5}$ do not all go through $P$.

**Proof.** Gröbner bases with a lot of manual help.

**Corollary.** No six partnered pairs through one point, so no point on ramification curve lies on more than 10 lines.

**Fact.** In each characteristic there is an example of a del Pezzo surface $X$ with 10 concurrent lines.
Del Pezzo surfaces of degree one

**Proposition (case $P$ off the ramification curve)**
Assume that $\text{char } k = 0$.
Let $Q_1, \ldots, Q_8$ be eight points in $\mathbb{P}^2$ in general position. Set
$L_1$ is the line through $Q_1$ and $Q_2$,
$L_2$ is the line through $Q_3$ and $Q_4$,
$C_1$ is the conic through $Q_1$, $Q_3$, $Q_5$, $Q_6$, and $Q_7$,
$C_2$ is the conic through $Q_1$, $Q_4$, $Q_5$, $Q_6$, and $Q_8$,
$C_3$ is the conic through $Q_2$, $Q_3$, $Q_5$, $Q_7$, and $Q_8$,
$C_4$ is the conic through $Q_2$, $Q_4$, $Q_6$, $Q_7$, and $Q_8$,
$D_1$ is the quartic through all points, singular at $Q_1$, $Q_7$, and $Q_8$
$D_2$ is the quartic through all points, singular at $Q_2$, $Q_5$, and $Q_6$
$D_3$ is the quartic through all points, singular at $Q_3$, $Q_6$, and $Q_8$
$D_4$ is the quartic through all points, singular at $Q_4$, $Q_5$, and $Q_7$.
There is no point that lies on all these curves.
Corollary.
No point in $X$ off the ramification curve lies on $>10$ lines.

Sketch of proof of Corollary.

- These 10 curves are a subset of a set of 11 (and even 12) curves that form a complete subgraph without any partnered pairs.
- The group $G$ acts transitively on the set of all sets of 11 such curves.
- If there are 11 concurrent lines on $X$, then (as before), there is a blow down $X \rightarrow \mathbb{P}^2$ such that 10 of these curves have images as described in the proposition.
- The proposition gives a contradiction.
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- The proposition gives a contradiction.
Sketch of proof of proposition.

Define

\[(\mathbb{P}^2)^9 \ni \Gamma = \{(P, Q_1, \ldots, Q_8) : Q_1, Q_2, \ldots, Q_8 \text{ not in general pos'n}\}, \]

\[(\mathbb{P}^2)^9 \ni \Delta = \{(P, Q_1, \ldots, Q_8) : \text{the curves in the prop'n contain } P\} \]

We will show \(\Delta \subset \Gamma\), or equivalently, \(Z := \Delta \cap (\mathbb{P}^2)^9 \setminus \Gamma) = \emptyset\).

The group \(\text{PGL}_3(k)\) acts on everything. After showing that for \((P, Q_1, \ldots, Q_8) \in Z\), no three of \(P, Q_1, Q_5, Q_6\) lie on a line, we may restrict to

\[(\mathbb{P}^2)^9 \ni P = \left\{(P, Q_1, \ldots, Q_8) : \begin{array}{c} P = [-1:0:1] \\ Q_1 = [1:0:1] \\ Q_5 = [0:1:1] \\ Q_6 = [0:-1:1] \end{array} \right\} \]

\(\Gamma' = \Gamma \cap P\)

\(\Delta' = \Delta \cap P\)

\(Z' = Z \cap P = \Delta' \cap (P \setminus \Gamma')\)
\[ P \sim = (P_2) \supset Y = \{ (Q_2, Q_3, Q_4, Q_7, Q_8) : P \in L_1, L_2, C_1, C_2, C_3 \} \]
\[ P \supset Y = \{ (Q_2, Q_3, Q_4, Q_7, Q_8) : P \in L_1, L_2, C_1, C_2, C_3 \} \]
\[ P \sim (P_2) \supset Y = \{(Q_2, Q_3, Q_4, Q_7, Q_8) : P \in L_1, L_2, C_1, C_2, C_3\} \]
\[
\neg = (P_2) \cup Y = \{ (Q_2, Q_3, Q_4, Q_7, Q_8) : P \in L_1, L_2, C_1, C_2, C_3 \}
\]
\[ Q_1 \rightarrow Q_2 \rightarrow Q_3 \rightarrow Q_4 \rightarrow Q_5 \rightarrow Q_6 \rightarrow Q_7 \rightarrow Q_8 \]

\[ L_1 \rightarrow C_1 \rightarrow C_2 \rightarrow C_3 \]

\[ P = (Q_2, Q_3, Q_4, Q_7, Q_8) \]

\[ L_5 \rightarrow \mu \rightarrow \lambda \rightarrow t \rightarrow u \]
\[ \sim \mathcal{Y} = (P_2, Q_3, Q_4, Q_7, Q_8) : P \in L_1, L_2, C_1, C_2, C_3] \]
\[ P \cong (\mathbb{P}^2)^5 \supseteq Y = \{(Q_2, Q_3, Q_4, Q_7, Q_8) : P \in L_1, L_2, C_1, C_2, C_3\} \]
\[ \mathbb{P} \cong (\mathbb{P}^2)^5 \supset Y = \{(Q_2, Q_3, Q_4, Q_7, Q_8) : \]
\[ P \in L_1, L_2, C_1, C_2, C_3 \} \]
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\[
\nu \in \lambda \cap \mu \cap t \\
\{Q_2, Q_3, Q_4, Q_7, Q_8\} = (P-2)^5 \cup Y \\
P \in L_1, L_2, C_1, C_2, C_3
\]
\[ \mathbb{P} \cong (\mathbb{P}^2)^5 \supset Y = \{(Q_2, Q_3, Q_4, Q_7, Q_8) : P \in L_1, L_2, C_1, C_2, C_3\} \]
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\( \mathbb{P} \cong (\mathbb{P}^2)^5 \supset Y = \{(Q_2, Q_3, Q_4, Q_7, Q_8) : P \in L_1, L_2, C_1, C_2, C_3\} \)
$\mathbb{P} \simeq (\mathbb{P}^2)^5 \supset Y = Q_3$

$\{ (Q_2, Q_3, Q_4, Q_7, Q_8) : P \in L_1, L_2, C_1, C_2, C_3 \}$
\[ \mathbb{P} \cong (\mathbb{P}^2)^5 \supset Y = Q_3 \]
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$P \cong (\mathbb{P}^2)^5 \supset Y = \{Q_3, (Q_2, Q_3, Q_4, Q_7, Q_8) : P \in L_1, L_2, C_1, C_2, C_3\}$
\[ \mathbb{P} \cong (\mathbb{P}^2)^5 \supset Y = \{ (Q_2, Q_3, Q_4, Q_7, Q_8) : P \in L_1, L_2, C_1, C_2, C_3 \} \]
\[ \mathbb{P} \cong (\mathbb{P}^2)^5 \supseteq Y = \{(Q_2, Q_3, Q_4, Q_7, Q_8) : P \in L_1, L_2, C_1, C_2, C_3\} \leftrightarrow L^5 \]

The extra requirement \( P \in C_4 \) yields a hypersurface in \( L^5(\lambda, \mu, \nu, t, u) \) that is a conic bundle over \( L^3(\lambda, \mu, \nu) \) with a section. Hence, it is birational to \( \mathbb{A}^4 \).
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In \( \mathbb{A}^4 \), the four conditions \( P \in D_i \) for \( 1 \leq i \leq 4 \) define a set that is contained in the set that describes \textbf{not} being in general position (at this point MAGMA is able to help out).

This proves the proposition.
\[ P \cong (\mathbb{P}^2)^5 \subset Y = \{(Q_2, Q_3, Q_4, Q_7, Q_8) : P \in L_1, L_2, C_1, C_2, C_3\} \leftrightarrow L^5 \]

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This proves the proposition.

**Proof of the theorem (case \( P \) off the ramification curve).**
The corollary already said that \( > 10 \) concurrent lines is impossible. There is a 2-dimensional family of examples with 10 concurrent lines.
Thank you