Log Curves

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Goal: understand a log version of this

Log Curves (Rosa Schwarz, 4-11-2020)

3. log curve
   Jesse \to log smooth + classification

4. vary log strk? 
   "stack log curves"

1. Divisorial log structure + examples

2. $\overline{M_{gn}}$ in $\overline{M_{gn}}$ and log strk.
§1. Log structures associated to divisors

Let $D \subset X$ be a NCD, then we define a log str on $X$ by

$$M(U) = \{ f \in \mathcal{O}_X(U) \mid \text{div } f \in D \}$$

with the inclusion $\mathcal{O}_X \supset M(U)$.

Examples

1. [Pim 3]

$X = \text{Spec}(k[x,y])$, $D$ given $xy = 0$

$N = \{ 0 \}$

the divisorial log str

$$M_X = \mathcal{O}_X \oplus \text{i}_x\text{-axis} \oplus N \oplus \text{i}_y\text{-axis} \oplus N$$

Log str assoc $N^2 \rightarrow k[x,y]$

$$\pi(x,y) \mapsto x^a y^b$$

2. $\text{Spec } k[N]$ divisor at $x = 0$

$\text{Spec}(k[x])$

3. [Margherita]

Toric log str are isom to divisorial log at the complement
How far can we push this? (Non-)examples

4. Trivial log st ≅ (X, Gx) is divisorial
5. (X, Gx) not divisorial if X = φ

6. \[ n \geq 1 \quad N \rightarrow h_n \quad \Rightarrow \quad N = N \oplus h^x_n \]

7. Pullback

Properties

Motto: Divisorial log st is "nice."

Lemma: Let D ⊆ X, SNC divisor. Then the divisorial log st on X makes X into an fs log scheme locally it has a chart modelled by an fs normal.

Proof: For x ∈ D, log st is trivial. For x ∈ D, Ox,x regular local ring, \( x_1, \ldots, x_d \) regular system of params. Then \( \text{D} \Rightarrow (x_1, \ldots, x_d) \Rightarrow \text{X, x} \)
Now define $\mathcal{M}_{g,n}$ by

Objects: $C \to S$ smooth proper
gem. fibres $C_g$ proj. curves genus $g$.

Morphisms: Pullback diagrams

For example $\mathcal{M}_{0,n}$

$\mathcal{M}_{1,2}$

and $\sigma_1, \ldots, \sigma_n: S \to S$ in disjoint sectors
To compactify $M_g, n \in M_{g, n}$, use crucial def by Deligne- Mumford:

\[ \xrightarrow{n \text{-marked}} \]

**Def:** A semi-stable curve of genus $g$ is morphism

\[ \pi : C \rightarrow S \]

proper flat

whose geometric fibres $C_s$ are reduced, canon curves s.t

1. Only ordinary double points as singularities, i.e. étale locally
   \[ \text{Spec} \left( \frac{k[x,y]}{(xy)} \right) \]

2. \[ \dim H^1(C_s, \mathcal{O}) = g \]

It is stable if in addition every non-singular rational component meets other component in more than 2 pts.

the curve had finitely many automorphisms
- genus 0 must have $\geq 3$ special pts
- genus 1 $\geq 21$ special pt

node $\xrightarrow{n \text{-marked}}$
Example:
\[ M_{0,1} \quad \rightarrow \quad \overline{M}_{1,2} \quad \sim \quad P' \]
\[ P' \setminus \{0,1,\infty\} \sim M_{0,1} \]

\[ \xrightarrow{1} \quad \xrightarrow{2} \quad \xrightarrow{3} \quad \xrightarrow{4} \]
\[ 1 \quad 2 \quad 3 \quad 4 \]
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\[ \overline{M}_{0,1} \sim P' \]

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Properties (cf end Jan's talk)

- $\overline{\text{M}}_{g,n}$ DM-stack
  - smooth proper $\dim 3g - 3 + n$
- "boundary" is a NCD
  - intuitively $\overline{\text{M}}_{g,n} \setminus \text{M}_{g,n}$
  - dual graphs + images
  - define via fitting ideal
- $\overline{\text{M}}_{g,n+1}$ as the universal curve over $\overline{\text{M}}_{g,n}$

Give $\overline{\text{M}}_{g,n}$ divisorial log str.
§3 Log curves

Def. Let $S$ be an fs log scheme. A log curve over $S$ is a log smooth and integral morphism $f : X \to S$ of fs log schemes such that every geometric fiber of $f$ is reduced and connected.

Examples

1. $x = \text{Spec} \left( \mathbb{C}[x, y]/(xy) \right)$

   Then $\Delta : \mathbb{N} \to \mathbb{N}^2$

   log curve

2. $[\text{Pin}]$

   $x = \text{Spec} \left( \mathbb{C}[\mathbb{N}]/(\mathbb{N}^2) \right)$

3. Smooth $f : X \to S$ trivial log str. (when geom conn + dim 1)
A log smooth morphism can have non-reduced fibers.

\[ P = \langle a, b, c \mid 2a + b = c \rangle \]

\[ \text{Spec}(\mathbb{Z}[P]) \rightarrow \text{Spec}(\mathbb{Z}[\mathbb{N}]) \]

\[ \mathbb{Z}[t^3] \rightarrow \mathbb{Z}[x, y, t]/(x^2y - t) \]

\[ \mathbb{Z}[x, y]/(xy) \]

But is log smooth, check as in Jesse's talk via jet procedures here and takes coherent germs \( \mathbb{N} \rightarrow P \).
How about any stable curve?

Suppose \( S = \text{Spec} (A) \) and \( f: X \to S \) is strict Henselian, \( h \) a stable curve \( \overline{M}_n \)

Around a node \( \xi_i \) in the closed fibre we can find an etale nbhd \( U_i \)

\[
\begin{align*}
M_i & \to U_i \xrightarrow{\psi_i} \text{Spec}(A[x,y,t]/(xy-t)) \\
L_i & \to \text{Spec}(A[t])
\end{align*}
\]

Then pull back log str

Write \( N \) for the divisorial log str at the marked pts

\[
\begin{align*}
M_X &= M_i \oplus S_X \\
M_S &= L_i \oplus S_T
\end{align*}
\]

CONSTRUCTION \( \Delta \)
\[ E_q \]

\[ \mathbb{N}^3 \leftarrow \mathbb{N}^2 \oplus_{N} \mathbb{N}^2 \]

\[ (a, b) \mapsto (a \cdot a, b) \]
Theorem: If $X$ is a complete closed, $f: X \to S$ log curve with $S = \text{Spec}(k)$, then

1. $X$ has at most ordinary double points.
2. Let $\{e_1, \ldots, e_l\}$ be the set of nodes and $\{s_1, \ldots, s_l\}$ be the disjoint points not equal to $e_i$.

$$M_{X/S} = \frac{M_X}{\langle \text{im: } f^*MS \to M_X \rangle} = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{N} \oplus \cdots \oplus \mathbb{N},$$

why $a \oplus \mathbb{Z}$ at a node?

$$\triangle \rightarrow \cdots$$

$$\overline{M}_{X/S} = \text{cohe}_\Delta = \mathbb{Z}$$

$IN^2 \rightarrow \mathbb{Z}$

$(a, b) \mapsto a - b$
Local description of log curves in families

Reference:
Fumiharu Kato, Log smooth deformation and models of log smooth curves.

To look for:
'Table of local descriptions of log structures of log curves.'
§ 4 Different log structures on a stable curve?

\[ E_0 \rightarrow E_0 \text{ as a divisor on } C \]

\[ \rightarrow \text{ always get a cdlg log str.} \]

whereas log str. associated to stable curve / \( \mathbb{A} \)

depends - complexity graph
- \( n \) nodes

Theorem log str. given on \( \mathbb{A} \) is basic
\[ X/S \text{ s. n-moder g., and } X/S \text{ log curve via } \mathbb{A} \]

Suppose \( X'/S' \) log curve \( \mathcal{O}' \)
\[ \xrightarrow{E} X \]
\[ \xrightarrow{\pi} S \]

Then \( \mathcal{A} \)
\[ X \xrightarrow{E_0} X \]
\[ \xrightarrow{\pi} S' \]
\[ \xrightarrow{\pi} S \]
\[ \text{in fs log sch} \]
Construction A is same as making horizontal maps strict.

Note: if you put divisorial log str on right hand side and make horizontal maps strict, you get the same log str on the left hand side as when you do construction A on the left hand side.