Linear algebra 2: Homework set 1

Due date: September 27 13:45

(H1.1). Show that the 2×2 matrix

$$\begin{pmatrix} 1 & x \\ x & 2 \end{pmatrix}$$

is a diagonalizable matrix over $\mathbb{R}$ for all $x \in \mathbb{R}$. Can you show the same if we replace $\mathbb{R}$ by $\mathbb{C}$?

(H1.2). Show that the matrix

$$A = \begin{pmatrix} 4 & 6 \\ -1 & -1 \end{pmatrix}$$

is diagonalizable, and give a basis of $\mathbb{R}^2$ consisting of eigenvectors of A. Now consider the system of differential equations

$$\begin{aligned} f' &= 4f - g \\ g' &= 6f - g. \end{aligned}$$

We can also write this as $(f', g') = (f, g)A$. If $\begin{pmatrix} a \\ b \end{pmatrix}$ is an eigenvector of A with eigenvalue λ, and we have functions f, g that satisfy the differential equations, then show that $h = af + bg$ satisfies the differential equation $h' = \lambda h$. Use this to solve the system of differential equations.

(H1.3). Find an element $a \in \mathbb{R}$ for which the matrix

$$A_a = \begin{pmatrix} a & -a & -1 \\ 0 & 1 & -1 \\ 8 & -5 & 1 \end{pmatrix}$$

is nilpotent. (A square matrix is nilpotent $A^k = 0$ for some integer $k \geq 1$.)

(H1.4). In this exercise we look at the properties of products of nilpotent matrices.

1. Give an example of a positive integer n and two nilpotent $n \times n$ matrices A and B for which AB is not nilpotent.
2. Show that AB is nilpotent for any nilpotent $n \times n$ matrices A and B that satisfy $AB = BA$.
3. Does there exist a positive integer n and two nilpotent $n \times n$ matrices A and B for which AB is invertible? (Give a proof for your answer.)