Linear algebra 2: Homework set 3

Due date: November 8 13:45

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(H3.1). Let V be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. Consider the subspace W of V spanned by the functions f_1, f_2, f_3 given by $f_1(x) = \cos(x), f_2(x) = \sin(x)$ and $f_3(x) = \sin(2x)$. For $i = 1, 2, 3$ consider $\phi_i \in W^*$ defined by $\phi_i(f) = f((i-1)\pi/4)$

1. Compute the 3×3-matrix $(\phi_i(f_j))$.
2. Deduce that f_1, f_2, f_3 is a basis for W and that ϕ_1, ϕ_2, ϕ_3 is a basis of W^*.
3. Show that there are $a, b, c \in \mathbb{R}$ so that all functions $f \in W$ satisfy
$$\int_0^\pi x^2 f(x)dx = af(0) + bf(\pi/4) + cf(\pi/2)$$
4. Give the basis of W that is dual to the basis ϕ_1, ϕ_2, ϕ_3 of W^*.

(H3.2). Let V, W be vector spaces and let $f\colon V \to W$ be a linear map. Suppose that the dual map $f^T\colon W^* \to V^*$ is the zero map. Show that f is the zero map.

(H3.3). Consider the vector space $\mathcal{C}([0,1])$ of continuous real-valued functions on the unit interval with norms $||\cdot||_1$ and $||\cdot||_2$ as defined in Example 7.7. For $n > 0$ define $g_n \in \mathcal{C}([0,1])$ by
$$g_n(x) = \begin{cases} \sqrt{n} & \text{if } 0 \leq x \leq 1/n; \\ 1/\sqrt{x} & \text{if } 1/n \leq x \leq 1. \end{cases}$$
Show that $||g_n||_2 \to \infty$ as $n \to \infty$, and that $||g_n||_1 \to 2$ as $n \to \infty$. Show that the two norms are not equivalent.

(H3.4). Let V be a *finite dimensional* vector space over a field K and let $b\colon V \times V^* \to K$ be a bilinear map. Show that there is an endomorphism f of V so that $b(v, \phi) = \phi(f(v))$ for all $v \in V$ and $\phi \in V^*$.