Topics in group theory: exercises
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Exercise 109. Let G be a finite group, let H be a subnormal subgroup of G, let p be a prime number, and let S be a Sylow-p-subgroup of G. Prove: $S \cap H$ is a Sylow-p-subgroup of H.

Exercise 110. Let G be a finite group and for each prime p let O_p be the largest normal p-subgroup of G. Show that the Fitting subgroup $F(G)$ of G is the direct sum of all O_p.

Exercise 111. Let G be a finite group. Show that $\Phi(G) \subset F(G)$. [Hint: use the Frattini argument.]

Exercise 112. Let G be a finite group. Show that $F(G/Z(G)) = F(G)/Z(G)$.

Exercise 113. Suppose that G is a finite solvable group. Show that $C_G(F(G)) \subset F(G)$.

Exercise 114. Let G be a finite group, with Fitting subgroup F. Prove that the group $\mathrm{Soc}(C_G(F)/Z(F))$ is a *perfect* semisimple group.

Exercise 115. Let G be a group. A *maximal abelian subgroup* of G is an abelian subgroup A of G such that A is the only abelian subgroup of G that contains A.

(a) Prove that the union of all maximal abelian subgroups of G equals G, and that their intersection equals $Z(G)$.

(b) Prove that each maximal abelian subgroup of G equals its own centralizer in G.

(c) Suppose that G is non-abelian. Prove that G has at least three maximal abelian subgroups. Do you know a group having exactly three of them?

Exercise 116. Let G be a group.

(a) Prove that G has a *maximal abelian normal subgroup*, i. e., an abelian normal subgroup A of G such that A is the only abelian normal subgroup of G that contains A.

(b) Suppose that G is nilpotent. Prove that each maximal abelian normal subgroup of G equals its own centralizer in G.

Exercise 117. Let G be a group, and write $\mathcal{S}$ for the set of subgroups of G. Define $\gamma\colon \mathcal{S} \to \mathcal{S}$ by $\gamma(H) = C_G(H)$. Prove the following:

(a) if $H, I \in \mathcal{S}$ satisfy $H \subset I$, then $\gamma(I) \subset \gamma(H)$;

(b) for all $H \in \mathcal{S}$ one has $H \subset \gamma^2(H)$;

(c) $\gamma = \gamma^3$.

Exercise 118. Let the notation be as in Exercise 117. Prove that there are infinitely many groups G, up to isomorphism, for which $\gamma^2 = \mathrm{id}_{\mathcal{S}}$.

Exercise 119. Let p be a prime number, let k be a finite field of characteristic p, let n a positive integer, and let G be a p-Sylow subgroup of $\mathrm{GL}(n, k)$. Prove that G is generated by its abelian normal subgroups.

Exercise 120. Let G be a finite group of even order $n > 2$. Show that G has a subgroup $H \neq G$ with $\#H \geq \sqrt[3]{n}$. [Hint: use the lemma we used in class to show Brauer-Fowler.]