Topics in group theory: exercises
Mathematisch Instituut, Universiteit Leiden, Fall 2012
Bart de Smit & Hendrik Lenstra

Exercise 133. Let H be a subgroup of a group G. Prove: one has $C_G(H) = \{1\}$ if and only if for each subgroup $I \subset G$ with $H \subset I$ one has $Z(I) = \{1\}$.

Exercise 134. (a) Let A, B be groups whose underlying sets are subsets of some set in such a way that the group operations of A and B coincide on $C = A \cap B$. Assume that C is a subgroup both of $Z(A)$ and of $Z(B)$. Construct a group D containing A and B as normal subgroups such that $[A, B] = \{1\}$ and $AB = D$. Prove also that such a group D is uniquely determined up to a unique isomorphism that is the identity on $A \cup B$.

Note. We call D the *central sum* of A and B over C.

(b) Let A be a subgroup of a group G, and let B be a subgroup of $C_G(A)$. Prove that AB is a subgroup of G, and that it is the central sum of A and B over $A \cap B$.

Exercise 135. Let G be a group, and let S be a subset of G.

(a) Prove that there is a finite normal subgroup of G that contains S if and only if S satisfies the following three conditions: (i) S is finite; (ii) each $x \in S$ has finite order; (iii) for each $x \in S$, the conjugacy class ${}^G x$ of x in G is finite.

(b) Prove: if (i), (ii), (iii) are satisfied, then there is a normal subgroup N of G containing G with $\#N \leq \prod_{x \in S} (\text{order}\, x)^{\#({}^G x)}$.

(c) Let T be a finite set, and let ω, $\gamma\colon T \to \mathbf{Z}_{>0}$ be two functions. Construct a group Γ and a map $f\colon T \to \Gamma$ such that for every $t \in T$ one has $\omega(t) = \text{order}(f(t))$ and $\gamma(t) = \#({}^G f(t))$ and such that the smallest normal subgroup of Γ containing $f(T)$ has order exactly $\prod_{t \in T} \omega(t)^{\gamma(t)}$.

Exercise 136. Let G be a group with $Z(G) = \{1\}$, with automorphism tower $G = G_0 \subset G_1 \subset G_2 \subset \ldots$, and put $G_\omega = \bigcup_{i=0}^{\infty} G_i$. Prove that the normalizer of G in G_ω equals G_1.

Exercise 137. Let S, T be subnormal subgroups of a finite group G, with $G = ST$. Prove: $(ST)^\infty = S^\infty T^\infty$. (See Exercise 121 for the notation $^\infty$.) (*Hint*: first do the case S, T are normal in G, then work by induction.) Is your proof also valid for infinite G?

Exercise 138. (a) Let G be a finite group, and let N, M be normal subgroups of G with G/N nilpotent. In class we proved that G has a nilpotent subgroup H satisfying (i) $G = HN$; (ii) $M = (H \cap M)(N \cap M)$. Prove that H can be chosen so as to satisfy in addition: (iii) each prime number dividing $\#H$ divides $\#(G/N)$.

(b) Construct an infinite group G with a normal subgroup N for which G/N is nilpotent, such that there does not exist a nilpotent subgroup $H \subset G$ with $G = HN$.

Exercise 139. (a) Let N be a normal subgroup of a group G, with G/N abelian. Does it follow that $G = HN$ for some abelian subgroup H of G? Give a proof or a counterexample.

(b) Answer the same question, with both occurrences of "abelian" replaced by "cyclic".

Exercise 140. Let N, M be normal subgroups of a finite group G, with G/N cyclic.

(a) Does it follow that G has a cyclic subgroup H satisfying $G = HN$ and $M = (H \cap M)(N \cap M)$? Give a proof or a counterexample.

(b) Suppose that $\gcd(\#G/(MN), \#MN/N) = 1$. Prove that there does exist a cyclic subgroup H of G satisfying $G = HN$ and $M = (H \cap M)(N \cap M)$.

Exercise 141. Let k be a finite field, let V be a finite-dimensional vector space over k, and let $m \in \mathbf{Z}$ satisfy $0 \le m \le \dim_k V$. Give a formula, in terms of $\#k$, $\dim_k V$, and m, for the number of m-dimensional subspaces of V.

Exercise 142. For a group G, write $\mathcal{L}(G)$ for the set of subgroups of G. We say that two groups G_0 and G_1 have *isomorphic subgroup lattices* if there is a bijection $\alpha\colon \mathcal{L}(G_0) \to \mathcal{L}(G_1)$ with the property that for any H, $I \in \mathcal{L}(G_0)$ one has: $H \subset I \Leftrightarrow \alpha(H) \subset \alpha(I)$.

Show that there is an abelian group G such that G and a non-abelian group of order 6 have isomorphic subgroup lattices.

Exercise 143. Let p and q be prime numbers with q dividing $p(p-1)$, and let C be a cyclic group of order q.

(a) Prove: there is a group homomorphism $f\colon C \to \mathbf{F}_p^*$ with $\sum_{\gamma \in C} f(\gamma) = 0$ (in $\mathbf{F}_p$).

(b) Let f be as in (a), let V be a vector space over $\mathbf{F}_p$, and let C act on V by $\gamma v = f(\gamma) \cdot v$ (for $\gamma \in C$, $v \in V$). We write G for the semidirect product $V \rtimes C$, taken with respect to this action, and we view V as a subgroup of G in the natural way. Prove: each element of G that is not in V has order q, and G is abelian if and only if $V = 0$ or $q = p$.

Exercise 144. Let p, q, C, f, G be as in Exercise 143, and let δ be a generator of C. You may assume the results of Exercise 143(b).

(a) We call a subset X of V a *coset* if there are a subspace $W \subset V$ and an element $v \in V$ such that $X = v + W$. Prove that for each subgroup $H \subset G$ with $H \not\subset V$ the set $X_H = \{w \in V : (w, \delta) \in H\}$ is a coset, and that the map $H \mapsto X_H$ is a bijection from the set of subgroups $H \subset G$ with $H \not\subset V$ to the set of cosets in V.

(b) Prove that there is an abelian group A of exponent p such that G and A have isomorphic subgroup lattices, as defined in Exercise 142.