Topics in group theory: exercises
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Exercise 13. Let k be a finite field, and denote by p its characteristic. Let $G = k^+ \rtimes k^*$ be the semidirect product of the additive group k^+ and the multiplicative group k^* of k, the action of the latter on the former being multiplication in k.

(a) Prove that G is isomorphic to the subgroup $\{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) : a \in k^*,\, b \in k\}$ of $\mathrm{Gl}(2,k)$.

(b) Prove that G has a normal Sylow p-subgroup.

(c) Let l be a prime number different from p. Prove that G has a cyclic Sylow l-subgroup. For which l is the Sylow l-subgroup normal in G?

Exercise 14. Let k and G be as in Exercise 13, and let n be a positive integer. Prove that G has a subgroup of order n if and only if there are positive integers n_1 and n_2 satisfying

$$n = n_1 n_2, \qquad n_1 \text{ divides } \#k, \qquad n_2 \text{ divides } \#k^*, \qquad n_1 \equiv 1 \bmod n_2.$$

Exercise 15. Suppose that n is a positive integer with the property that every finite group of order divisible by n has a subgroup of order n. Prove that n is a prime power. (*Hint*: use Exercise 14.)

Exercise 16. For a group G acting on a set X, we write $X^G = \{x \in X : \text{ for all } \sigma \in G \text{ one has } \sigma x = x\}$. In class we used several times that if p is a prime number and G is a p-group acting on a finite set X, one has $\#X^G \equiv \#X \bmod p$. The present exercise is a converse.

Let G be a finite group with $\#G > 1$, and let a, b, $n \in \mathbf{Z}$, $n > 1$. Suppose that there does *not* exist a finite G-set X with $\#X \equiv a \bmod n$ and $\#X^G \equiv b \bmod n$. Prove that there exists a prime number p dividing n such that G is a p-group and $a \not\equiv b \bmod p$.

Exercise 17. Classify the groups of order 42, supplying all details. Where does the cyclic group of order 42 appear in your classification? And the dihedral group D_{21}? And the group $\mathbf{F}_7 \rtimes \mathbf{F}_7^*$ from Exercise 13?

Exercise 18. Let p be a prime number, G a finite group, and S a Sylow p-subgroup of G.

(a) Prove: if N a normal subgroup of G, then $S \cap N$ is a Sylow p-subgroup of N, and the image of S in G/N is a Sylow p-subgroup of G/N.

(b) Prove: if H is a subgroup of G, then there is a conjugate T of S in G such that $T \cap H$ is a Sylow p-subgroup of H. Can one always take $T = S$? Give a proof or a counterexample.

Exercise 19. Let p be a prime number. Prove: if G is a p-group, and $N \subset G$ is a normal subgroup of order greater than 1, then $N \cap Z(G) \neq \{1\}$.

Exercise 20. Prove that any group of order 1001 is cyclic.

Exercise 21. A *normal tower* of a group G is a sequence $(u_i)_{i=0}^r$ of subgroups u_i of G for which r is a non-negative integer, each u_{i-1} is a normal subgroup of u_i (for $0 < i \leq r$), and $u_0 = \{1\}$, $u_r = G$. A group is called *solvable* if it has a normal tower $(u_i)_{i=0}^r$ for which every group u_i/u_{i-1} is abelian. A group G is called *nilpotent* if it has a normal tower $(u_i)_{i=0}^r$ such that for each i, $0 \leq i < r$, the subgroup u_i is normal in G and u_{i+1}/u_i is contained in the center of G/u_i.

(a) Let G be a group, and let N be a normal subgroup of G. Prove: G is solvable if and only if both N and G/N are solvable and if and only if each subgroup of G is solvable.

(b) Prove that each subgroup and each factor group of a nilpotent group is nilpotent.

(c) Prove that each nilpotent group is solvable, and give an example of a solvable group that is not nilpotent.

Exercise 22. Let G be a group. The *derived series* of G is the sequence $(G^{(n)})_{n=0}^{\infty}$ of subgroups of G that is inductively defined by $G^{(0)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$.

(a) Prove that each $G^{(n)}$ is a characteristic subgroup of G.

(b) Prove: G is solvable if and only if there exists n with $G^{(n)} = \{1\}$.

Exercise 23. Let G be a group, and let N, M be abelian normal subgroups of G.

(a) Prove: $NM = \{xy : x \in N, y \in M\}$ is a nilpotent normal subgroup of G.

(b) Is NM necessarily abelian? Give a proof or a counterexample.

Exercise 24. Let G be a group. The *upper central series* of G is the sequence $(Z_n(G))_{n=0}^{\infty}$ of normal subgroups of G that is inductively defined by $Z_0(G) = \{1\}$ and $Z_{n+1}(G)/Z_n(G) = Z(G/Z_n(G))$, where the center of a group H is denoted by $Z(H)$.

Let G be a group, and let N, M be normal subgroups of G.

(a) Prove: for all non-negative integers n, m one has $Z_n(N) \cap Z_m(M) \subset Z_{n+m}(NM)$.

(b) Prove: if N and M are nilpotent, then NM is a nilpotent normal subgroup of G.