Topics in group theory: exercises
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Exercise 25. Show that a finite group is nilpotent if and only if for every pair x, y of group elements of coprime order we have $xy = yx$.

Exercise 26. For each of the four possible actions of the cyclic group C_2 of order 2 on the cyclic group C_8 of order 8 determine the nilpotency class of the semidirect product $C_8 \rtimes C_2$.

Exercise 27. Show that a p-group of order p^n with $n \geq 2$ has nilpotency class at most $n-1$. Show that the dihedral group of order 2^n has nilpotency class $n-1$ when $n \geq 2$.

Exercise 28. Let p be a prime number and $n \geq 1$. Consider the finite ring $R = \mathbb{F}_p[x]/(x^n)$. Let $\langle 1+x \rangle$ be the subgroup of the unit group of R generated by $1+x$. Compute the order and the nilpotency class of $R \rtimes \langle 1+x \rangle$.

Exercise 29. Let (G, X) and (H, Y) be permutation groups. Assume that Y is finite and that H acts transitively on Y. Show that there is a natural group isomorphism $(G \wr H)^{\mathrm{ab}} \cong G^{\mathrm{ab}} \times H^{\mathrm{ab}}$.

Exercise 30. Let C be a cyclic group of prime order p. Show that the n-fold wreath product $G = C \wr C \wr \cdots \wr C$ can be generated by n elements of order p. Show that G cannot be generated by fewer than n elements.

Exercise 31. Let (G, X) and (H, Y) and (I, Z) be permutation groups. Show that there is a natural isomorphism of permutation groups between $((G \wr H) \wr I, (X \times Y) \times Z)$ and $(G \wr (H \wr I), X \times (Y \times Z))$.

Exercise 32. Let G be a group that acts transitively on a set X. For a subset B of X show that the following are equivalent:

(1) The set B is non-empty and for every $g \in G$ the sets gB and B are either equal or disjoint.
(2) There is a surjective map of G-sets $f\colon X \to Y$ and an element $y \in Y$ such that $B = f^{-1}(y)$.
(3) There is an element $x \in B$ and a subgroup H of G that contains the stabilizer of x in G such that $B = Hx$.

When these conditions hold we say that B is a block of X.

Exercise 33. Let $G = \mathbb{Z}$ act by translation on $X = \mathbb{Z}$. What are the blocks of X?

Exercise 34. Let F be a finite field and let G be the group of permutations of $X = F$ which are of the form $x \mapsto ax + b$ with $a \in F^*$ and $b \in F$. Show that the action of G on X is 2-transitive, i.e., for all $x, y, x', y' \in F$ with $x \neq y$ and $x' \neq y'$ there is an element $g \in G$ with $gx = x'$ and $gy = y'$. Deduce that the action of G on X is primitive.

Exercise 35. Let l be a prime number, and suppose that $q \in \mathbb{Z}$ satisfies $q \equiv 1 \bmod l$. Moreover, if $l = 2$, assume that $q \equiv 1 \bmod 4$. Show that

$$\mathrm{ord}_l(q^n - 1) = \mathrm{ord}_l(n) + \mathrm{ord}_l(q - 1).$$

Exercise 36. In this problem we consider the $3 \times 3 \times 3$ Rubik's cube. By a move on the cube we mean that the cube is disassembled (the 8 corner pieces and 12 edge pieces are taken out) and put together again in some fashion. These moves form a group that acts on the set of all configurations of the cube. Show that this group of moves is isomorphic to $(C_2 \wr S_{12}) \times (C_3 \wr S_8)$. By a legal move we mean one where one rotates a faces, so that no disassembly is necessary. Find a strict subgroup that contains all the legal moves.