Topics in group theory: exercises
Mathematisch Instituut, Universiteit Leiden, Fall 2012
Bart de Smit & Hendrik Lenstra

Exercise 61. Let p be a prime number, and let n be a non-negative integer.

(a) Prove: every group of order p^n is cyclic if and only if $n \le 1$.

(b) Prove: every group of order p^n is abelian if and only if $n \le 2$.

Exercise 62. (a) Let G be a group. By a *G-set* or *left G-set* we mean a set X equipped with an action of G on X. A *right G-set* is a set X equipped with a map $X \times G \to X$, $(x, \sigma) \mapsto x\sigma$, with the property that for all $x \in X$, σ, $\tau \in G$ one has $x1 = x$ and $(x\sigma)\tau = x(\sigma\tau)$. Prove that each left G-set X can be turned into a right G-set by defining $x\sigma = \sigma^{-1}x$, and that every right G-set arises in this way.

(b) Let G, H be groups. By a *G-H-biset* we mean a set X that is both a left G-set and a right H-set, with the property that for all $\sigma \in G$, $x \in X$, $\rho \in H$ one has $(\sigma x)\rho = \sigma(x\rho)$. Prove that every $G \times H$-set X can be turned into a G-H-biset by putting $\sigma x = (\sigma, 1)x$ and $x\rho = (1, \rho^{-1})x$, for $\sigma \in G$, $x \in X$, $\rho \in H$, and that every G-H-biset arises in this way.

Exercise 63. Let G be a group. A G-set X is called *free* if for each $x \in X$ the stabilizer of x in G equals $\{1\}$, and it is called *regular* if it is both transitive and free. A regular G-set is also called a *G-torsor*.

(a) Show that for each group G there is a G-torsor, and that any two G-torsors are G-isomorphic.

(b) Show that the group of G-automorphisms of any G-torsor is isomorphic to G. To which extent is your isomorphism independent of choices?

Exercise 64. Let P be a Platonic solid. By a *flag* of P we mean a triple (v, e, f) where f, e, v are a face of P, an edge of f, and one of the endpoints of e, respectively. Let $\mathcal{F}$ be the set of flags of P, and denote by G the symmetry group of P. Explain that the natural action of G on $\mathcal{F}$ makes $\mathcal{F}$ into a G-torsor.

Exercise 65. (a) Let G, H be groups. By a *G-H-bitorsor* we mean a G-H-biset (see Exercise 62(b)) that is regular both as a G-set (see Exercise 63) and as an H-set (cf. Exercise 62(a)). Prove: a G-H-bitorsor exists if and only if $G \cong H$.

(b) Let G be a group, and let X, Y be G-G-bitorsors. By a *G-G-isomorphism* $X \to Y$ we mean a bijection $f\colon X \to Y$ with the property that for all $\sigma \in G$, $x \in X$ we have $\sigma f(x) = f(\sigma x)$ and $(f(x))\sigma = f(x\sigma)$; if such a map exists, we say that X and Y are *G-G-isomorphic.* A *G-G-automorphism* of X is a G-G-isomorphism $X \to X$.

Prove that the group of G-G-automorphisms of X is isomorphic to the center $Z(G)$ of G.

Exercise 66. (a) Let N be a group. Show that left and right multiplication by elements of N makes N into a N-N-bitorsor, as defined in Exercise 65.

(b) Let G be a group and let $N \subset G$ be a normal subgroup. Show that left and right multiplication by elements of N makes every coset γN of N in G into a N-N-bitorsor.

(c) Give an example of a group G, a normal subgroup N of G, and an element $\gamma \in G$, such that the N-N-bitorsors N from (a) and γN from (b) are not N-N-isomorphic.

Exercise 67. Let G be a group. We define $\operatorname{Out} G = (\operatorname{Aut} G)/\operatorname{Inn} G$, the group of automorphisms of G modulo the normal subgroup of inner automorphisms of G; one often calls $\operatorname{Out} G$ the group of *outer automorphisms* of G.

(a) Let X, Y be G-G-bitorsors. Call two elements of $X \times Y$ *equivalent* if there are $x \in X$, $\sigma \in G$, $y \in Y$ such that the first element equals $(x\sigma, y)$ and the second element equals $(x, \sigma y)$. Prove that this is an equivalence relation on $X \times Y$, and that the set $X * Y$ (say) of equivalence classes is a G-G-torsor through the actions $\sigma[x, y] = [\sigma x, y]$, $[x, y]\sigma = [x, y\sigma]$; here $\sigma \in G$, $x \in X$, $y \in Y$, and $[x, y]$ denotes the equivalence class to which (x, y) belongs.

(b) Prove that the operation $*$ from (a) turns the set of G-G-isomorphism classes (cf. Exercise 65(b)) of G-G-bitorsors into a group, and that this group is isomorphic to the group $\operatorname{Out} G$.

Exercise 68. (This exercise combines Exercises 66(b) and 67.) Let G be a group, and let $N \subset G$ be a normal subgroup.

(a) Prove that for every γ, $\delta \in G$ the N-N-bitorsor $\gamma N * \delta N$, with $*$ as in Exercise 67(a), is N-N-isomorphic to the N-N-bitorsor $\gamma\delta N$.

(b) Show that (a) and Exercise 67(b) give rise to a group homomorphism $G/N \to \operatorname{Out} N$. Can you describe this group homomorphism in a direct way?

Exercise 69. Let G be a group. In class we defined a *G-group* to be a group A together with a group homomorphism $G \to \operatorname{Aut} A$, denoted $\sigma \mapsto (a \mapsto {}^{\sigma}a)$.

Let A be a G-group. If a set X is both a G-set and an A-set, then the three actions (of G on A, of G on X, and of A on X) are called *compatible* if for all $\sigma \in G$, $a \in A$, $x \in X$ one has $\sigma(ax) = ({}^{\sigma}a)(\sigma x)$.

Let X be a set. Prove that equipping X with an action of G and an action of A such that the three actions are compatible, is equivalent to equipping it with an action of $A \rtimes G$; here the semidirect product $A \rtimes G$ is formed with respect to the given G-group structure on A.

Exercise 70. (a) Let G be a group, let A be a G-group (see Exercise 69), and let X be a group that is both a G-group and an A-group. Assume that the compatibility condition of Exercise 69 is satisfied. Prove that, in a natural way, X is a $A \rtimes G$-group and $X \rtimes A$ is a G-group, and that the semidirect products $X \rtimes (A \rtimes G)$ and $(X \rtimes A) \rtimes G$ are naturally isomorphic; so it may be written $X \rtimes A \rtimes G$.

(b) With X, A, G equal to, respectively, the additive group of $\mathbf{F}_4$, the multiplicative group $\mathbf{F}_4^*$, and the group $\operatorname{Aut}\mathbf{F}_4$ of field automorphisms of $\mathbf{F}_4$, define actions as in (a) such that the compatibility conditions are satisfied and such that the group $\mathbf{F}_4 \rtimes \mathbf{F}_4^* \rtimes \operatorname{Aut}\mathbf{F}_4$ is isomorphic to the symmetric group S_4.

Exercise 71. Let G be a group, and let A be a G-module, i.e., a G-group that is *abelian*; we write A additively. A 1-*cocycle* or *crossed homomorphism* from G to A is a map $c\colon G \to A$ with the property that for all σ, $\tau \in G$ one has $c(\sigma\tau) = c(\sigma) + \sigma c(\tau)$. The set of 1-cocycles from G to A forms, with point-wise addition, an abelian group, which is denoted $Z^1(G, A)$. A 1-*coboundary* from G to A is a map $c\colon G \to A$ for which there exists $b \in A$ such that for all $\sigma \in G$ one has $c(\sigma) = b - \sigma b$. The set of 1-coboundaries from G to A is a subgroup of $Z^1(G, A)$, which is denoted $B^1(G, A)$. Finally, the *first cohomology group* $H^1(G, A)$ of G with coefficients in A is defined by $H^1(G, A) = Z^1(G, A)/B^1(G, A)$.

(a) Suppose G is an infinite cyclic group, with generator ρ. Prove that for every $a \in A$ there is exactly one 1-cocycle $G \to A$ that maps ρ to a, and that one has $Z^1(G, A) \cong A$. To which description of $H^1(G, A)$ does this give rise?

(b) Suppose now that $G = \langle\rho\rangle$ is a *finite* cyclic group, and let m be its order. Prove: $H^1(G, A) \cong \{a \in A : \sum_{i=0}^{m-1} \rho^i a = 0\}/\{b - \rho b : b \in A\}$.

Exercise 72. Let G be a group of order 2, and let A be the group $\mathbf{Z}/4\mathbf{Z}$. Prove that the number of G-module structures on A equals 2, and compute the groups $Z^1(G, A)$, $B^1(G, A)$, $H^1(G, A)$ defined in Exercise 71 for both of them.