Topics in group theory: exercises

Mathematisch Instituut, Universiteit Leiden, Fall 2012

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Exercise 73. Let $n \geq 4$. Show that A_n is a characteristic subgroup of S_n and that the restriction map $\mathrm{Aut}(S_n) \to \mathrm{Aut}(A_n)$ is injective. Show that it also surjective for $n = 4$. Can you show surjectivity for $n > 4$ too (this may be hard)?

Exercise 74. Let $G = (\mathbf{Z}/2\mathbf{Z}) \times (\mathbf{Z}/8\mathbf{Z})$. Show that G has a unique cyclic subgroup C of order 4 so that G/C is also cyclic. What is the order of $\mathrm{Aut}_{C,G/C}(G)$? Give an element of order 2 in $\mathrm{Aut}_{C,G/C}(G)$.

Exercise 75. Let D_n be the dihedral group of order $2n$. How many automorphisms of D_8 fix the subgroup D_4 pointwise? Give such an automorphism that is not the identity.

Exercise 76. Let G be a group and let $\mathbf{Z}[G]$ be the group ring of G over $\mathbf{Z}$. Let $I(G) \subset \mathbf{Z}[G]$ be the kernel of the homomorphism $\mathbf{Z}[G] \to \mathbf{Z}$ that sends all elements of G to 1. For any G-module A give an isomorphism $Z^1(G, A) \to \mathrm{Hom}_G(I(G), A)$. Can you describe the image of $B^1(G, A)$?

Exercise 77. Let G be a group. For a short exact sequence of G-modules

$$0 \to A \to B \to C \to 0$$

show that there is an exact sequence

$$0 \to H^0(G, A) \to H^0(G, B) \to H^0(G, C) \to H^1(G, A) \to H^1(G, B) \to H^1(G, C).$$

Exercise 78. Show that the sequence in the previous exercise is functorial in the sense that a morphism of short exact sequences gives rise to a morphism of long exact sequences.

Exercise 79. Let G be a group and let A be an abelian group. Show that we have a G-module structure on $A^G = \mathrm{Map}(G, A)$ defined by $(gf)(h) = f(hg)$ for $f \in A^G$ and $g, h \in G$. For $c \in Z^1(G, A^G)$ define $a\colon G \to A$ by $a(g) = c(g)(1)$. Show that $ga - a = c(g)$ for all $g \in G$. Deduce that $H^1(G, A^G) = 0$.

Exercise 80. With the notation of the previous exercise, prove that $H^2(G, A^G) = 0$.

Exercise 81. Let G be a group of finite order n, let A be a G module, and let $c \in Z^2(G, A)$. Define $f \in A^G = \mathrm{Map}(G, A)$ by $f(g) = \sum_{h \in G} c(g, h)$. Putting $(\partial f)(g, h) = f(gh) - f(g) - gf(h)$, show that $(\partial f)(g, h) = -nc(g, h)$. Deduce that $n \cdot H^2(G, A) = 0$.

Exercise 82. Let G be a finite group, and let A be a G-module which is finitely generated as an abelian group. Show that $H^1(G, A)$ and $H^2(G, A)$ are finite.

Exercise 83. Let G be a finite group, and let A be a G-module which is uniquely divisible, i.e., for all $a \in A$ and $n \geq 1$ there is a unique element $b \in A$ with $a = nb$. Show that $H^1(G, A) = H^2(G, A) = 0$.

Exercise 84. Let $\mathbf{Z}/2\mathbf{Z}$ act trivially on $\mathbf{Z}$. Show that $H^2(\mathbf{Z}/2\mathbf{Z}, \mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$. What are the two extensions of $\mathbf{Z}/2\mathbf{Z}$ by $\mathbf{Z}$?