

hog Curves

Rosa  
Schwarz





# Log Curves

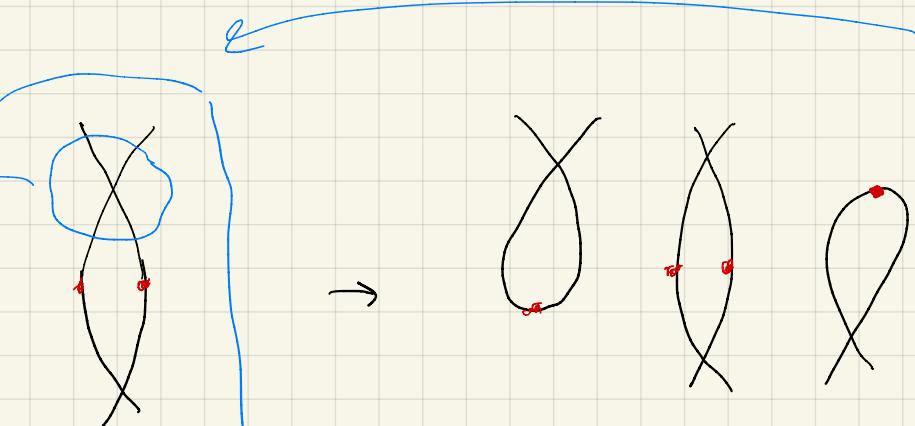
(Rosa Schwarz, 4-11-2020)

1

Goal: understand a log version of this

③ log curve

Jesse =>  
log smooth  
+ classification

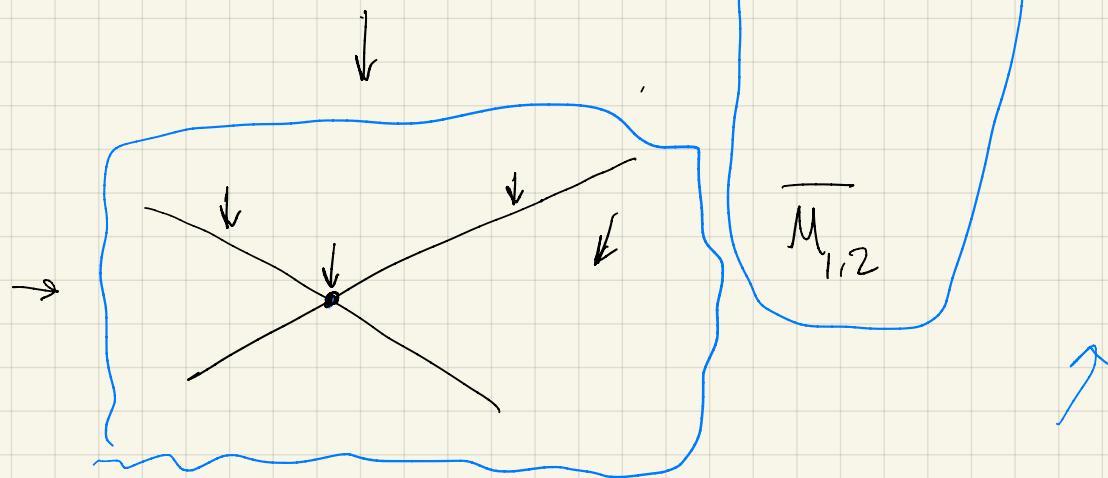


② Mag.

$\overline{M}_{\text{gn}}$

and  
log str.

④ vary  
log str?



① Divisorial log structure  
+ examples

"stack log curves"

# §1. Log structures associated to divisors

Def Let  $D \subset X$  be a NCD, then we define a log str on  $X$  by

$$\mathcal{M}(U) = \{ f \in \mathcal{O}_X(U) \mid f|_{U \cap D} \in (\mathcal{O}_X^*)^3 \}$$

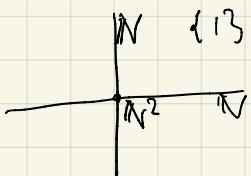
with the inclusion  $\alpha: \mathcal{M} \rightarrow \mathcal{O}_X$

$$\mathcal{O}_X^*$$

Examples

(1) [Pim]

$X = \text{Spec}(k[x, y])$ ,  $D$  given  $xy=0$



the divisorial log str

$$\mathcal{M}_X = \mathcal{O}_X^* \oplus_{\text{ix-axis}} \mathbb{N} \oplus_{\text{iy-axis}} \mathbb{N}$$

log str asso

$$\begin{aligned} \mathbb{N}^2 &\rightarrow k[x, y] \\ (a, b) &\mapsto x^a y^b \end{aligned}$$

Normal crossings divisor:

eff Cartier Div st  
 $\forall p \in D \exists$  étale nbh  $U$   
 of  $p$  st

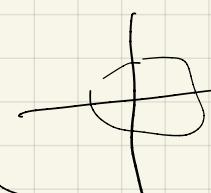
$D \times_X U$  is strict normal crossings

SNC

eff Cartier Div st  $\forall p \in D$   
 -  $(\mathcal{O}_{X,p})^*$  regular

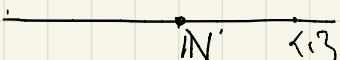
-  $\exists$  reg syst  $x_1, \dots, x_d$   
 st  $D$  is given by  
 $x_1 \cdots x_d$

$$xy=0$$



(2)  $\text{Spec } k[\mathbb{N}]$

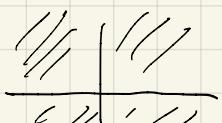
divis or at  $x=0$   
 $\text{Spec}(k[\mathbb{N}])$



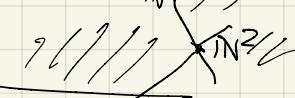
(3) [Maegherita]

Toric log str are isom to divisorial log at the complement  
 of the torus

isom to divisorial log at the complement



$\dashrightarrow$



(3)

How far can we push this? (Non-)examples

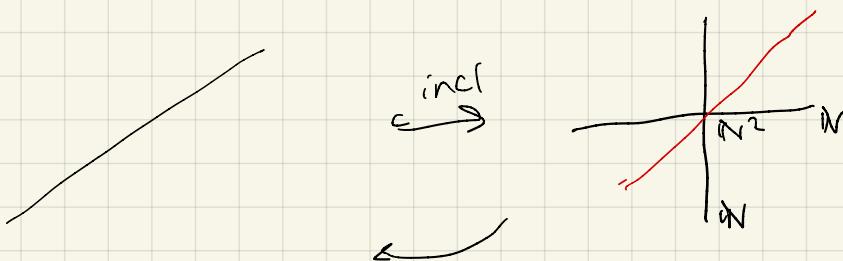
(4) Trivial log str  $(X, G_X^*)$  is divisorial

(5)  $(X, G_X)$  not divisorial if  $X = \emptyset$

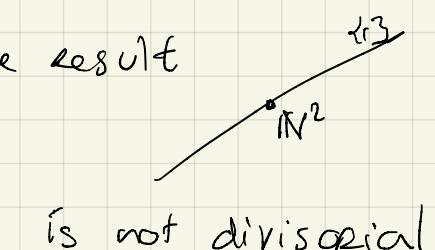
(6)  $n \geq 1$   $N \rightarrow h$   $\rightarrow M = N \oplus h^*$   
 $\downarrow \cong$   $O^n$

$N^h$

(7) Pullback



The result



is not divisorial

Properties

Motto: Divisorial log str is "nice".

Lemma: let  $D \subset X$  SNC divisor. Then the divisorial log str on  $X$  makes  $X$  into an fs log scheme.

state locally it has a chart modelled by an fs toric

Proof: For  $x \notin D$ , log str is trivial

$P=1/3$

log system para.

For  $x \in D$ :  $O_{X,x}$  regular local ring,  $x_1, \dots, x_d$  reg. ~~reg. s.t.~~  $D = (x_1 \cdots x_k = 0)$

$$(n_1, \dots, n_d) \mapsto O_{X,x} \otimes \prod_i x_i^{n_i}$$

□

## §2 $M_{g,n}$ and $\overline{M}_{g,n}$

(4)

Now define  $M_{g,n} \xrightarrow{\pi} (\text{Sch})$  by

OBJECTS :  $C \xrightarrow{\pi} S$  smooth proper

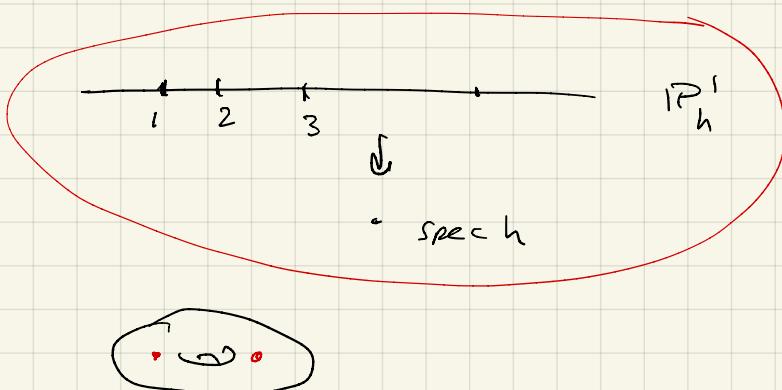
geom. fibres  $C_s$  proj. curves genus  $g$ .

and  $\sigma_1, \dots, \sigma_n : S \rightarrow C$   
n disjoint sections

MORPHISMS: Pullback diagrams

$$\begin{array}{ccc} C_1 & \longrightarrow & C_2 \\ \downarrow \pi & & \downarrow \\ S_1 & \longrightarrow & S_2 \end{array}$$

For example  $M_{0,n}$



$M_{1,2}$

To compactify  $M_{g,n} \subset \overline{M}_{g,n}$ , use crucial def by Deligne-Mumford:

$n$ -marked  
Def: A  $\mathbb{R}$  semi-stable curve of genus  $g$  is morphism

$$\pi : C \longrightarrow S \quad \text{proper flat}$$

whose geometric fibres  $C_s$  are reduced, conn curves  $\checkmark$

- ① only ordinary double points as singularities,  
i.e. étale locally

$$\text{Spec}\left(\frac{\mathbb{R}[x,y]}{(xy)}\right)$$

- ②  $\dim H^1(C_s, \mathcal{O}) = g$

It is stable if in addition every non-singular rational component meets other components in more than 2 pts.

the curve had finitely many automorphisms

- genus 0 must have  $\geq 3$  special points
- genus 1  $\geq 1$  special pt

node  
marking

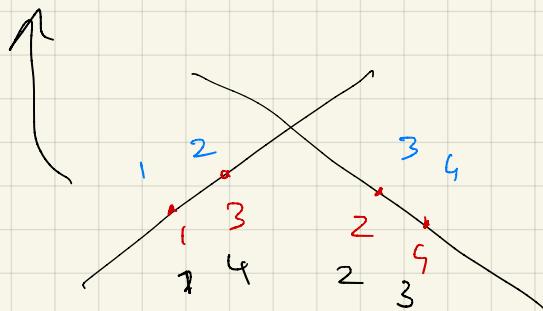
(5)

Example:

$$\overline{M}_{0,4}$$

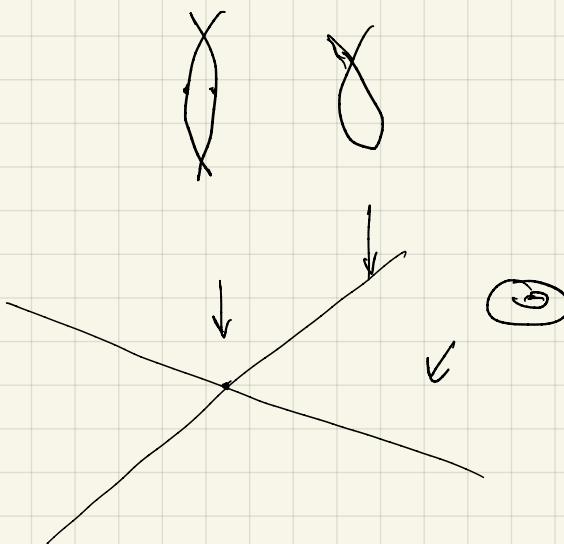
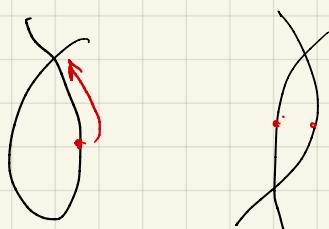
$$\overset{?}{\longrightarrow} \mathbb{P}^1$$

$$\mathbb{P}^1 \setminus \{0, 1, \infty\} \cong M_{0,4}$$



$$\rightsquigarrow \overline{M}_{0,4} \cong \mathbb{P}^1$$

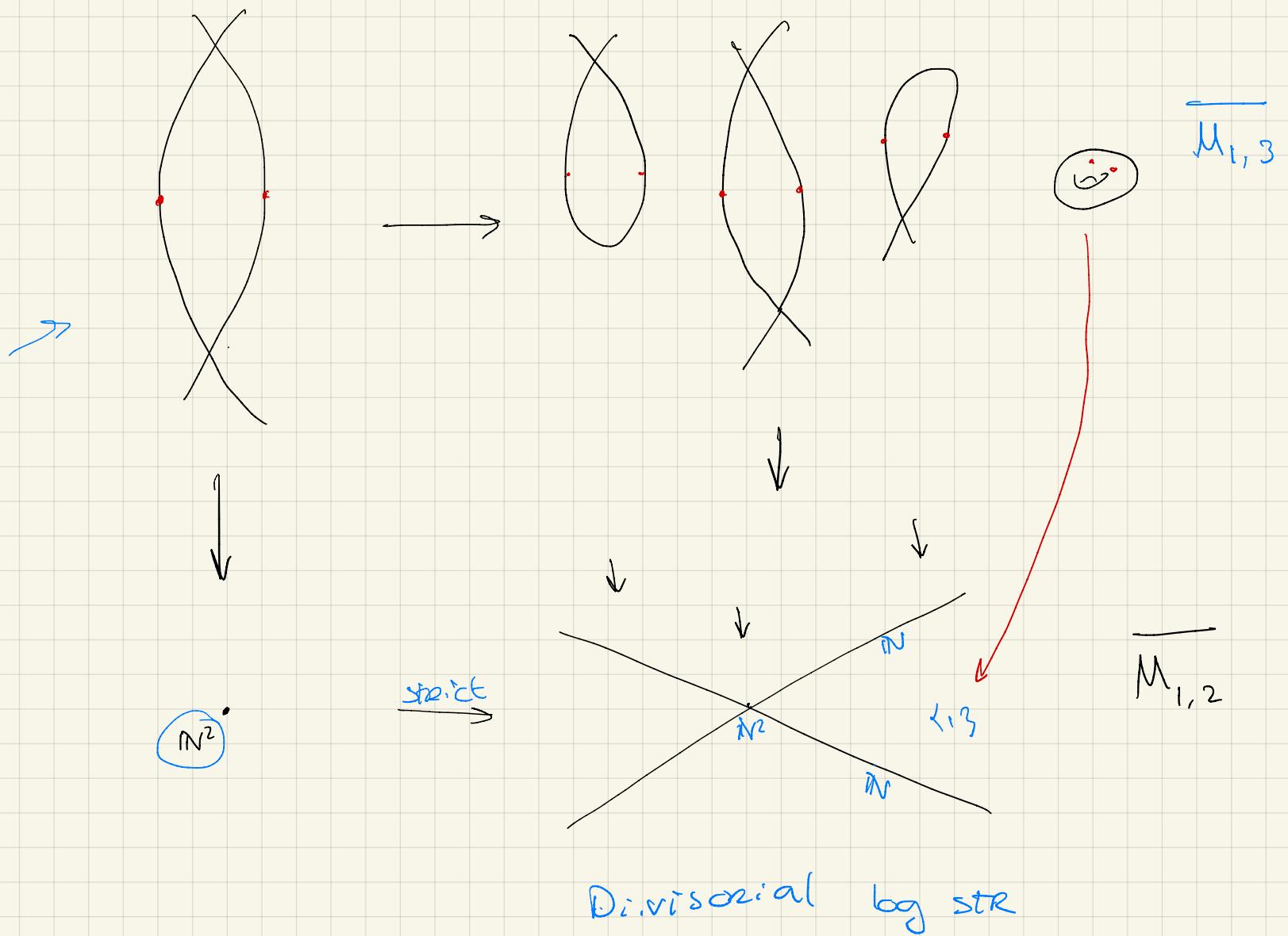
$$\overline{M}_{1,2}$$



# Properties (cf end Jan's talk)

- $\overline{M}_{g,n}$  DM-stack
  - ↳ smooth proper  $\dim \mathcal{Z}g - 3 + n$
- "boundary" is a NCD
  - ↳ intuitively  $\overline{M}_{g,n} \setminus M_{g,n}$
  - ↳ dual graphs + images
  - ↳ define via fitting ideal
- $\overline{M}_{g,n+1}$  as the universal curve over  $\overline{M}_g$

→ Give  $\overline{M}_{g,n}$  divisorial log str.



### §3 Log curves

(9)

Def let  $S$  be an fs log scheme. A log curve over  $S$  is a log smooth  
and integral morphism  $f: X \rightarrow S$  of fs log schemes st  
every geometric fiber of  $f$  is reduced and connected curve

#### Examples

1

$$x = \text{Spec } h[x, y]/(xy)$$

Then  $\Delta: \mathbb{N} \rightarrow \mathbb{N}^2$

$\downarrow$

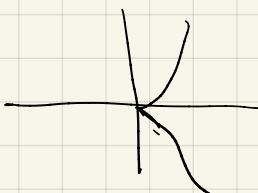
$\circ: \mathbb{N} \xleftarrow{\quad} \mathbb{N} \rightarrow h_a$   
 $a \mapsto O_a$

(log str  $\begin{matrix} \mathbb{N}^2 \\ (a, b) \end{matrix} \rightarrow h[x, y]/(xy) \end{matrix}$ )

log curve

#### 2 [Pim]

$$x = \text{Spec}(h[\mathbb{N} \setminus \{3\}])$$



3 Smooth  $f: X \rightarrow S$

trivial log str

(when geom conn + dim 1)

ex 4 A log smooth morphism can have non-reduced fibers

$$\boxed{P = \langle a, b, c \mid 2a + b = c \rangle}$$

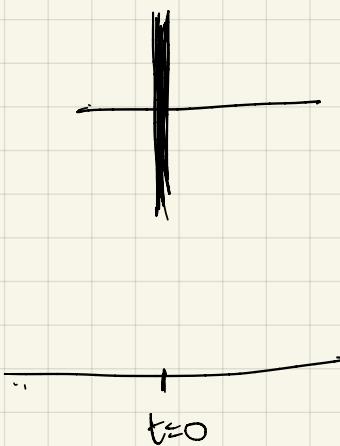
$$\begin{matrix} N & \rightarrow & P \\ 1 & \mapsto & c \end{matrix}$$

$$\rightarrow \text{Spec}(h[P]) \rightarrow \text{Spec}(h[N])$$

$$\begin{matrix} h[t] \\ t \end{matrix} \rightarrow \begin{matrix} h[x, y, t] \\ + \end{matrix} / (x^2y - t)$$

$$h[x, y] / (xy)$$

↑  
but is log sm  
(check as in Jesse's  
talk via  
herz +  
tors colors  
grp  $N \rightarrow P$ )



How about any stable curve?

Suppose  $S = \text{Spec}(A)$   
strict Henselian

and  $f: X \rightarrow S$

$h[[t]]$ ,  $h$

stable curve  $\overline{M}_{g,n}$

Around a node  $r_i$  in the closed fibre we can find an étale nbhd  $U_i$ :  
 $R_i \hookrightarrow R_E$

$$M_i - U_i \xrightarrow{\psi_i} \text{Spec}\left(A[x, y, t]/(xy-t)\right)$$



$$L_i - S \xrightarrow{\varphi_i} \text{Spec}(A[t])$$



Then pull back log str

Write  $N$  for the divisorial log str at the marked pts

$$M_X = M_1 \oplus_{\mathcal{O}_X^*} \dots \oplus_{\mathcal{O}_X^*} M_t \oplus_{\mathcal{O}_X^*} N$$

$$M_S = L_1 \oplus_{\mathcal{O}_S^*} \dots \oplus_{\mathcal{O}_S^*} L_t$$

$$\begin{array}{ccc} \leftarrow & \mathbb{N}^2 & \rightarrow A[x, y, t] \\ & (a, b) & \mapsto x^a y^b \\ & \uparrow \Delta & \end{array}$$
  

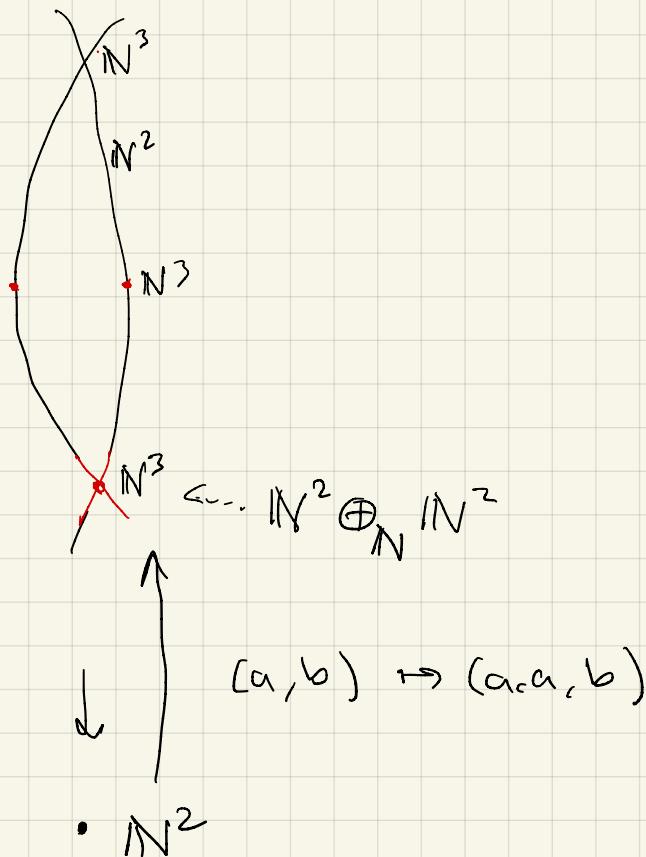
$$\leftarrow \mathbb{N} \rightarrow A[t]$$

↓      ↓

start here ↑  
then pullback

CONSTRUCTION A

Ex 8



Theorem:  $h$  sep closed,  $f: X \rightarrow S$  log curve with  $S = \text{Spec}(h)$  (13)

①  $X$  has at most ordinary double pts

② let  $\{r_1, \dots, r_l\}$  set of nodes and  $\exists s_1, \dots, s_n$  disj pts not equal to the  $r_i$

$$\overline{M}_{X/S} = M_X / (\text{im}: f^* M_S \rightarrow M_X) = \mathbb{Z}_{r_1} \oplus \dots \oplus \mathbb{Z}_{r_l} \oplus \mathbb{N}_{s_1} \oplus \dots \oplus \mathbb{N}_{s_n}$$

why a  $\mathbb{Z}$  at a node?

$$\overline{M}_{X/S} = \text{coher } \Delta = \mathbb{Z}$$

$$\begin{aligned} N^2 &\rightarrow \mathbb{Z} \\ (a, b) &\mapsto a - b \end{aligned}$$

## Local description of log curves in families

Reference:

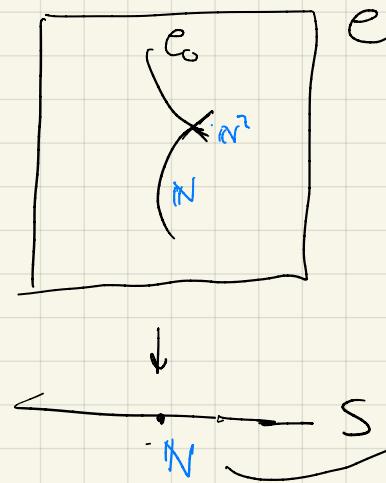
Fumiharu Kato, Log smooth deformation and moduli of log smooth curves.

↳ look for

'Table of local descriptions of log structures of log curves'.

## § 4 Different log structures on a stable curve?

(15)



$e_0$  as a divisor on  $C$

→ always get a sh! log str.

whereas log str asso to stable curve / (A)  
depends - complexity graph  
-  $n \neq$  nodes

Theorem log str given in (A) is basic

$X/S$  s. n-mader g, and  $X'/S'$  log curve via (A)  
Suppose  $X'/S'$  log curve  $\xrightarrow{D}$

$$\begin{array}{ccc} X' & \xrightarrow{b} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{a} & S \end{array}$$

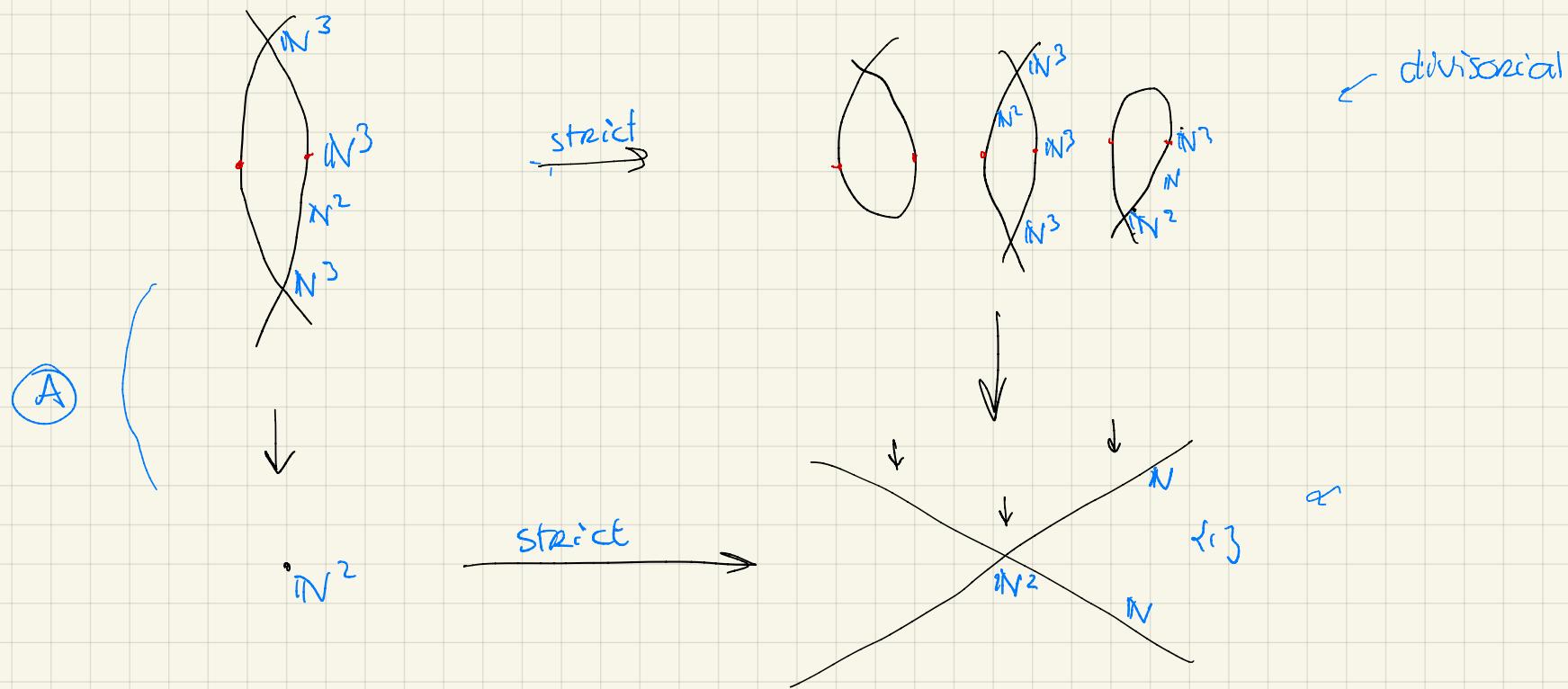
Then ?

$$\begin{array}{ccc} X & \xrightarrow{?b} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{?a} & S \end{array}$$

in fs log sch

Construction A is same as making horizontal maps strict

(16)



Note: if you put divisorial log str on right hand side  
and make horizontal maps strict,

you get the same log str on the left hand side as when you  
do construction (A) on the left hand side.

