

Log Properties

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Log smooth, étale
and differentials

Recall A log scheme (X, M) is fine if it is coherent and integral: étale locally on X ,

$$P_X \longrightarrow \mathcal{O}_X$$

fin. gen.
& integral

such that $(P_X)^a \cong M$

Def Let $i: (X, M) \rightarrow (Y, N)$ morph. of fine log sch. Say i is a closed immersion if $X \rightarrow Y$ is so, and $i^*N \rightarrow M$ surjective. If $i^*N \xrightarrow{\sim} M$ isomorphism, then i exact closed immersion.

Def Let $f: (X, M) \rightarrow (Y, N)$ be morph. of fine log sch. Then f is formally smooth/formally étale if

$$\begin{array}{ccc} (T', L') & \xrightarrow{\quad} & (X, M) \\ i \downarrow & g \dashrightarrow & \downarrow f \\ \text{exact cl. imm. } (T, L) & \xrightarrow{\quad} & (Y, N) \end{array}$$

& $T' \subseteq T$ given by
ideal I s.t. $I^2 = 0$

g unique

We say f is smooth/étale if it is so formally,
and $X \rightarrow Y$ is locally of finite presentation.

Ex If $f: (X, M) \rightarrow (Y, N)$ strict ($f^*N \xrightarrow{\sim} M$)
then f is log smooth/log étale
iff $X \rightarrow Y$ is smooth/étale.

$$\varphi^{gp} : Q^{gp} \rightarrow P^{gp} \quad Q \rightarrow Q^{gp}$$

Prop Let P, Q be f.g. integral monoids, $\varphi : Q \rightarrow P$,
 R a ring such that $\ker(\varphi^{gp})$ and $\text{coker}(\varphi^{gp})$ _{torsion}
are finite, of order invertible in R . Take

$$X = \text{Spec}(R[P]) \quad Y = \text{Spec}(R[Q])$$

with canonical log str. M and N . Then
 $(X, M) \rightarrow (Y, N)$ is smooth/étale.

Proof $I \hookrightarrow O_T^* \subset L$

$$x \mapsto 1+x$$

using $I^2 = 0$

$$\begin{array}{ccc}
\textcircled{1} & L \longrightarrow L/I = L' & \left. \begin{array}{l} P \rightarrow L \\ \Updownarrow \\ (T, L) \rightarrow (X, M) \end{array} \right\} \\
& \downarrow \text{cartesian.} & \\
L^{gp} & \longrightarrow L^{gp}/I = (L')^{gp} &
\end{array}$$

$$\textcircled{2} \quad \begin{array}{ccc}
(L')^{gp} & \longleftarrow & P^{gp} \\
\uparrow^{gp} & \swarrow & \uparrow \varphi^{gp} \\
Q^{gp} & \longleftarrow &
\end{array}$$

Thm Let $f: (X, M) \rightarrow (Y, N)$ be morph. of fine log sch. Assume chart $Q_Y \rightarrow N$. TFAE:

(i) f is $\overset{\text{log}}{\checkmark}$ smooth / étale

$$\begin{array}{ccc} Q_X & \xrightarrow{\varphi} & P_X \\ \downarrow & & \downarrow \\ f^*N & \longrightarrow & M \end{array}$$

(ii) étale locally on X , there exists a chart $(P_X \rightarrow M, Q_Y \rightarrow N, \varphi: Q \rightarrow P)$ of f such that

(a) $\ker(\varphi^g)$ and $\text{coker}(\varphi^g)$ _{torsion}

are finite of order invertible on X .

$$\begin{array}{ccc} (b) \quad X & \xrightarrow{\quad} & Y \times_{\text{Spec } \mathbb{Z}[Q]} \text{Spec } \mathbb{Z}[P] \longrightarrow \text{Spec } \mathbb{Z}[P] \\ f \quad \swarrow & & \downarrow \quad \downarrow \\ & Y & \longrightarrow \text{Spec } \mathbb{Z}[Q] \end{array}$$

(\dashrightarrow) is étale.

Example (Log smooth curve)

$$\bullet \longrightarrow o \in A^1$$

$$X = \text{Spec } k[x,y]/(xy) \quad Y = \text{Spec } k$$

$$M: P = N^2 \rightarrow k[x,y]$$

" " $(a,b) \mapsto x^a y^b$

$$(N^2 \oplus O_X^*)^u$$

$$N: Q = N \rightarrow k$$

" " $a \mapsto o^a$

$$\begin{cases} N \otimes k^* \\ (a, u) \mapsto o^a \cdot u \end{cases}$$

$f: (X, M) \rightarrow (Y, N)$ induced by

$$\begin{array}{ccc} \Delta: N \rightarrow N^2 & & \\ a \mapsto (a, a) & & \\ f^{-1}N = \underline{N \otimes k^*} & \xrightarrow{\quad (a, u) \quad} & M \\ f^{-1}\beta \downarrow & & \downarrow \alpha \\ k = O_Y & \longrightarrow & O_X^{(xy)^{a \cdot u}} \\ O^a \cdot u & \xrightarrow{\quad (xy)^{a \cdot u} \quad} & O^a \cdot u \end{array}$$

To show: f is log smooth. $\text{coker} \cong \mathbb{Z}$.

① $\Delta^{\otimes P}: \mathbb{Z} \rightarrow \mathbb{Z}^2: a \mapsto (a, a)$ $\ker = \text{coker}_{\text{tors}} = (0)$

② $X \rightarrow \text{Spec } k \times_{\text{id}, (t=0)} \text{Spec } k[t] \xrightarrow{\quad \text{Spec } k[x,y] = X \quad} \text{Spec } k[t] \quad (t=xy)$

Example (Toroidal embedding)

sch./lc of finite locally finite type
 $(X, M) \rightarrow (\text{Speck}, k^*)$

Smooth



toroidal embedding.

Log differentials

$\alpha: M \rightarrow \Omega_X^1$

Def Let $f: (X, M) \rightarrow (Y, N)$ be morph. of log schemes. Then the Ω_X^1 -module of log differentials $\omega_{X/Y}^1$ is the quotient of

$$\Omega_{X/Y}^1 \oplus (\Omega_X^1 \otimes_{\mathbb{Z}} M^{\text{gp}}) \quad \alpha(a) = \alpha(b)$$

by relations of local sections $a, b \in M$

$$(0, 1 \otimes a)$$

$$(i) \quad (d\alpha(a), 0) = (0, \alpha(a) \otimes a) \quad \text{for } a \in M$$

$$(ii) \quad (0, 1 \otimes a) = 0 \quad \text{for } a \in \underline{\text{im}(f^{-1}N \rightarrow M)}$$

$$\textcircled{1} \quad (0, 1 \otimes a) \xrightarrow{\text{think}} d\log(a) \stackrel{\text{def}}{=} \frac{da}{\alpha(a)} \quad \begin{matrix} \alpha(a) \\ \alpha(a) \end{matrix}$$

$\text{(ii)} \quad \tilde{d}a = a \cdot d\log(a)$

$$\textcircled{2} \quad d\log: M \rightarrow \omega_{X/Y}^1$$

$$fg \mapsto \frac{d(fg)}{fg} = \frac{df}{f} \cdot \frac{dg}{g}$$

$$\textcircled{3} \quad \Omega_X^1 \otimes_{\mathbb{Z}} M^{\text{gp}} \longrightarrow \omega_{X/Y}^1 \quad \text{surjective.}$$

$$a \otimes b \longmapsto a \cdot d\log(b)$$

$$df = f \cdot d\log(f)$$

Example

④ $X = \text{Spec } R[P]$ $Y = \text{Spec } R[Q]$
 $(X, M) \rightarrow (Y, N)$ induced $\varphi: Q \rightarrow P$

Then

$$\mathcal{O}_X \otimes_{\mathbb{Z}} (P^{\text{gp}} / \text{im } \varphi^{\text{gp}}) \xrightarrow{\sim} \omega_{X/Y}^{x/y}$$
$$a \otimes b \longmapsto a \cdot d \log(b)$$

Example Let k be field with $\text{char}(k) \neq 2$.

Consider

$$\begin{array}{ccc} A'_k & \xrightarrow{t=s^2} & A'_k \\ (s) & & (t) \end{array}$$

Usual $\Omega'_{X/Y}$ is given by free $k[x]$ -module with gen. dx . So,

$$\left. \begin{array}{ccc} \Omega'_{A'_k} & \longrightarrow & \Omega'_{A'_k} \\ dt & \longmapsto & 2s \cdot ds \end{array} \right\} \begin{array}{l} \text{coker has} \\ \text{Support at } s=0 \end{array}$$

Let $D = \{0\}$ divisor on A'_k

$$M = \{f \in \mathcal{O}_X \mid f \text{ is invertible outside } D\}$$

$$M \xrightarrow{\sim} \mathcal{O}_X \quad \begin{matrix} f(x) \neq 0 \\ f(x \neq 0) \neq 0 \end{matrix}$$

Nar, $x \in M$ so $\boxed{\omega'_{A'_k}}$ $\ni d\log(x) = \frac{dx}{x}$

(is free $k[x]$ -module with gen $\frac{dx}{x}$).

$$\left. \begin{array}{ccc} \omega'_{A'_k} & \longrightarrow & \omega'_{A'_k} \\ \frac{dt}{t} & \longmapsto & 2 \frac{ds}{s} \end{array} \right\} \text{isomorphism}$$

Remark As for schemes, if
 $f: (X, M) \rightarrow (Y, N)$ is smooth,
then $\omega'_{X/Y}$ is locally free of
finite type.