

# Basic definitions and examples of log schemes

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## 1 Basic definitions

**Definition 1.1.** A *monoid* is a set with a binary operator that's commutative, associative and contains a unit.

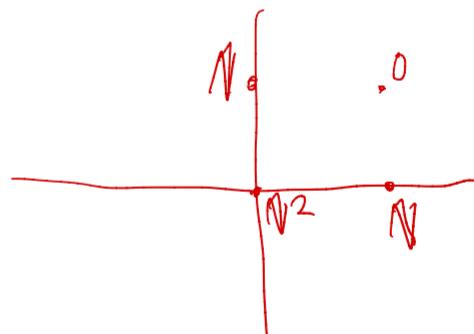
**Definition 1.2.** The characteristic of a monoid  $M$  is  $\overline{M} = M/M^*$ .

*Example 1.3.*

$G$  a group,  $\overline{G} = \{1\}$

$R$  a ring,  $\overline{R} = \{\text{principal ideals}\}$  if  $R$  is a domain

$(\mathbb{N}, +)$   $\overline{\mathbb{N}} = \mathbb{N}$



**Definition 1.4.** A *log scheme*  $X$  is a scheme  $\underline{X}$  together with a sheaf of monoids  $M_X$  on the topological space of  $\underline{X}$  and a monoid morphism  $\alpha : M_X \rightarrow \mathcal{O}_X^*$  such that  $\alpha : \alpha^{-1}\mathcal{O}_X^* \rightarrow \mathcal{O}_X^*$  is an isomorphism.

*Remark 1.5.* This implies  $M_X^* = \alpha^{-1}\mathcal{O}_X^*$ .

*Example 1.6.*

1)  $M_X = \mathcal{O}_X^*$ , trivial log structure on  $\underline{X}$

2)  $\underline{X} = \text{Spec } K[x, y]$ ,  $D$  the subscheme given by  $xy=0$   
 $M_X^{(u)} = \{f \in \mathcal{O}_X^{(u)} \mid f \text{ invertible outside of } D\}$

$M_X \rightarrow \mathcal{O}_X$  is inclusion

2.5)  $U = X - D$ ,  $M_X = j_* \mathcal{O}_U$   
 $j : U \rightarrow X$

3)  $\underline{X} = \text{Spec } K[x, y]$   $M_X = \mathcal{O}_X^* \oplus i_{x-\text{axis}, X}^! \mathbb{N}$

$\mathbb{N}^2 \rightarrow K[x, y]$   
 $(a, b) \mapsto x^a y^b$  gives

$M_X^! = \mathbb{N}^2 \oplus \mathcal{O}_X^* \rightarrow \mathcal{O}_X$

$M_X = M_X^! \oplus_{\alpha^{-1}(\mathcal{O}_X^*)} \mathcal{O}_X^*$

$\oplus_{i_{y-\text{axis}, X}^!} \mathbb{N}$

4)  $\underline{Y} = \text{Spec } K$

$\mathbb{N} \rightarrow K$   
 $n \mapsto 0^n$

$M_Y = \mathbb{N} \oplus K^*$

**Definition 1.7.** A pre-log structure on a scheme  $\underline{X}$  is sheaf of monoids  $M'_X$  on the topological space of  $\underline{X}$  and a monoid morphism  $\alpha : M'_X \rightarrow \mathcal{O}_X$ . The log structure associated to it, is  $M'_X \oplus_{\alpha^{-1}(\mathcal{O}_X^*)} \mathcal{O}_X^*$ .

**Definition 1.8.** Let  $X = (\underline{X}, M), Y = (\underline{Y}, N)$  be two log schemes, let  $f : \underline{X} \rightarrow \underline{Y}$  be a morphism of schemes. The pullback  $f^*N$  is the log structure associated to  $f^{-1}N$  with the structure map  $f^{-1}N \rightarrow f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$ . The pushforward  $f_*M$  is the fiber product  $f_*^{\text{sheaf}} M \times_{f_*\mathcal{O}_X} \mathcal{O}_Y$ .

**Lemma 1.9.**

$$\text{Hom}(f^*N, M) \xrightarrow{\sim} \text{Hom}(N, f_*M) \xrightarrow{\sim} \left\{ \begin{array}{l} f : X \rightarrow Y \text{ extending} \\ \perp \end{array} \right\}$$

**Definition 1.10.** A morphism of log schemes  $X = (\underline{X}, M) \rightarrow Y = (\underline{Y}, N)$  is a morphism of schemes  $f : \underline{X} \rightarrow \underline{Y}$  together with a map  $h : f^{-1}N \rightarrow M$  s.t. we get a commutative diagram

$$\begin{array}{ccc} f^1N & \rightarrow & M \\ \downarrow & & \downarrow \\ f^{-1}\mathcal{O}_Y & \rightarrow & \mathcal{O}_X \end{array}$$

*Example 1.11.*

1)  $\mathcal{O}_X^* \rightarrow \mathcal{O}_X$  is the initial object in log struct/ $\underline{X}$     3)  $Z = \text{Spec } k[X, Y]/(XY)$ , log structure

2)  $X : \begin{array}{c} N \\ \downarrow \\ \mathbb{N}^0 \end{array} \rightarrow \begin{array}{c} \mathbb{N} \\ \times \end{array} \text{ Spec } k[N]$

$$\mathcal{O}_{X,p} \rightarrow k$$

$$\begin{array}{ccc} N^2 \oplus \Gamma(X, \mathcal{O}_X^*) & \leftarrow & N \oplus k^* \\ (q_1 q_2, u) & \leftarrow & (q, u) \end{array}$$

by ~~pulling back along~~  
~~base changing~~  $Z \rightarrow X$

$$\begin{array}{c} N \\ \downarrow \\ N^2 \oplus N \\ \downarrow \\ \mathbb{N} \end{array}$$

$$m_{X,p} = \mathcal{O}_{X,p}^*$$

$$\begin{array}{ccc} 0 & \longrightarrow & N \\ \mathcal{O}_{X,p}^* & \longleftarrow & N \oplus k^* \\ (xy)^n \cdot u & \longleftarrow & (n, u) \end{array}$$

## 2 Charts

*Example 2.1.*  $P$  a monoid,  $X$  a scheme, map  $P \rightarrow \Gamma(X, \mathcal{O}_X)$ .

$$\text{induces } \underline{P} \rightarrow \mathcal{O}_X \text{ induces a log structure on } X$$

subex.  $\mathbb{N}^2 \rightarrow K[X, Y]$

$$N \rightarrow K$$

$$\mathbb{Z}[P] = \bigoplus_{p \in P} \mathbb{Z} \cdot p \quad \mathbb{Z}[N] \cong \mathbb{Z}[x]$$

$$i \cdot n \mapsto x^n$$

*Example 2.2.*  $P$  a monoid,  $\underline{X} = \text{Spec } \mathbb{Z}[P]$  a scheme, canonical log structure on  $\text{Spec } \mathbb{Z}[P]$ , denoted by  $P \rightarrow \text{Spec } \mathbb{Z}[P]$ .

$$P \rightarrow \Gamma(X, \mathcal{O}_X) \rightsquigarrow \mathbb{Z}[P] \rightarrow \Gamma(X, \mathcal{O}_X) \rightsquigarrow X \rightarrow \text{Spec } \mathbb{Z}[P]$$

**Definition 2.3.** A morphism  $X \rightarrow Y$  of log schemes is strict if  $f^*M_Y \rightarrow M_X$  is an isomorphism.

**Lemma 2.4.**  $X$  a log schier

$$\mathrm{Hom}(X, P \rightarrow \mathrm{Spec} \mathbb{Z}[P]) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{mon}}(P, \Gamma(X, \mathcal{M}_X))$$

*Proof.*

$$\rightarrow: f: X \rightarrow \mathrm{Spec} \mathbb{Z}[P] \text{ gives } f^* P_{\mathbb{Z}[P]} = P_X \rightarrow \mathcal{M}_X$$

Taking global sections gives  $P \rightarrow \Gamma(X, \mathcal{M}_X)$

$$\leftarrow: f: P \rightarrow \Gamma(X, \mathcal{M}_X), \quad P_X \rightarrow \mathcal{M}_X, \text{ take associated log structure}$$

$$P_{\mathbb{Z}[P]}$$

**Definition 2.5.** A chart of  $X$  modelled on a monoid  $P$  is a strict morphism  $X \rightarrow (P \rightarrow \mathrm{Spec} \mathbb{Z}[P])$ .

*Example 2.6.* Log curve has a chart modelled on  $\mathbb{N}^2$ , with  $\mathbb{N}^2 \rightarrow k[x, y]/(xy)$  given by  $(a, b) \mapsto x^a y^b$ .

$$k[x, y]/(xy)$$

### 3 Back to monoids

$P$  a monoid

**Definition 3.1.** The group envelope  $P^{\mathrm{gp}}$  is  $\{pq^{-1} \mid p, q \in P\}/\sim$  where  $pq^{-1} \sim p'q'^{-1} \Leftrightarrow \exists r \in P, rpq' = rp'q$ .

**Definition 3.2.**  $P$  is integral if  $P \rightarrow P^{\mathrm{gp}}$  is injective, equivalently  $P$  is cancellative:  $ab = ac \implies b = c$ .

$$p \mapsto p \uparrow$$

$$\mathbb{N}^{\mathrm{gp}} = \mathbb{Z}$$

$$G \text{ a group, } G^{\mathrm{gp}} = G$$

$$R \text{ a ring, } R^{\mathrm{gp}} = \{1\}$$

**Definition 3.3.**  $P$  is saturated if for  $\alpha \in P^{\text{gp}}$  with  $\alpha^k \in \text{im}(P \rightarrow P^{\text{gp}})$ , have  $\alpha \in \text{im } P$ .

*Non-Example 3.4.*  $P = \mathbb{N} \setminus \{1\}$

$$k[P] = k[x, y]/(y^2 - x^3)$$

$$\begin{aligned} 2 &\mapsto x \\ 3 &\mapsto y \end{aligned}$$



**Definition 3.5.**  $P$  is fine if fin. generated and integral,  $P$  is fs if fine and saturated.

**Lemma 3.6.** The characteristic  $\overline{P}$  is  $P/P^*$ ; if  $P$  is fs then  $\overline{P}^{\text{gp}} = \mathbb{Z}^n$ .

#### 4 Back to log schemes

**Definition 4.1.** A log scheme  $X$  is integral (fine/fs) if étale locally, it has a chart modelled over an integral (fine/fs) monoid.

**Proposition 4.2.** In characteristic 0, the sheaf of monoids of an fs log scheme  $X$  splits étale locally as  $\overline{M_X} \oplus O_X^*$ .

$$0 \rightarrow O_X^* \rightarrow M_X \rightarrow \overline{M_X} \rightarrow 0$$