Log seminar - exercise class

October 28, 2020

1. Verify that the three definitions 2, 2.5 and 3 of the first lecture give isomorphic log schemes.
2. Given a divisor D on a scheme X, define the notion of the corresponding divisorial log structure. Show for a few of the examples we have treated that they are in fact divisorial log structures.
3. Describe the toric variety corresponding to the fan generated by the cones $\langle e_1, e_1 + e_2 \rangle$ and $\langle e_1 + e_2, e_2 \rangle$.
4. Fact: a toric variety coming from a (rational, strongly convex, polyhedral) cone is smooth if and only if the cone is generated by a subset of a basis of the lattice. Use this to give a non-smooth toric variety.
5. Show that all toric varieties, with the natural log structure, are log smooth over the point with trivial log structure.
6. Analogously to Jan's talk, we can define the stack $\mathcal{M}_{g,n}$, consisting of genus g curves $C \to S$ with n disjoint sections $p_1, \ldots, p_n : S \to C$. Show that $\mathcal{M}_{0,3}$ is $\operatorname{Spec} \mathbb{Z}$, and $\mathcal{C}_{0,3}$, the universal curve over $\mathcal{M}_{0,3}$, is $\mathbb{P}^1$ with sections $0, 1, \infty$.
7. Show that $\mathcal{M}_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, and give the universal curve $\mathcal{C}_{0,4}$ and its four sections. Can you guess what $\overline{\mathcal{M}}_{0,4}$ and $\overline{\mathcal{C}}_{0,4}$ are?
8. If you know what a stable curve is: prove that a genus 0 stable curve (such a curve necessarily has ≥ 3 marked points) has no non-trivial automorphisms. (Hint: look at the dual graph). (This exercise actually implies that $\overline{\mathcal{M}}_{0,n}$ is a scheme by Theorem 3.19e of https://www.math.uni-bonn.de/people/schmitt/ModCurves/Script.pdf)
9. Show that $\operatorname{dlog} : M \to \Omega_X$ is a morphism of monoids (where Ω_X is a monoid under addition).

10. Compute the sheaf of differentials and the sheaf of log differentials in the following cases:

    (a) $\mathbb{A}^1_s \to \mathbb{A}^1_t$ given by $t = s^2$, with divisorial log structure respectively from $s = 0$ and $t = 0$.

    (b) $\mathbb{A}^1_{x,y} \to \mathbb{A}^1_{u,v}$ given by $(u, v) = (x, xy)$, with divisorial log structure respectively from $xy = 0$ and $uv = 0$.

    (c) $\mathbb{A}^2_{x,y} \to \mathbb{A}^1_t$ given by $t = xy$, with divisorial log structure respectively from $xy = 0$ and $t = 0$. (Note that the basechange to $t = 0$ gives the log smooth curve treated by Jesse.)