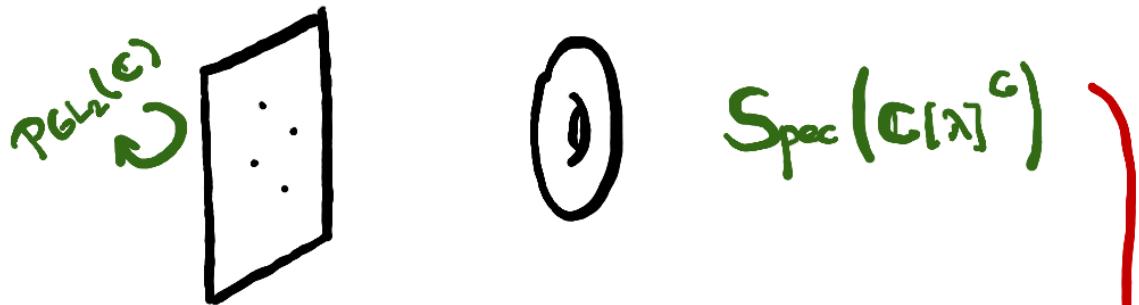


# Log-Geometry Seminar

Q: Can we classify all smooth proj. curves  $X_C$  of genus  $g$ ?

Genus 1: Any ell curve  $X_C$  can be written as:

a)  $\mathbb{P}^1 \xleftarrow{2:1} X : y^2 = x(x-1)(x-\lambda)$



b) quotient of  $\mathbb{C}$  by period lattice

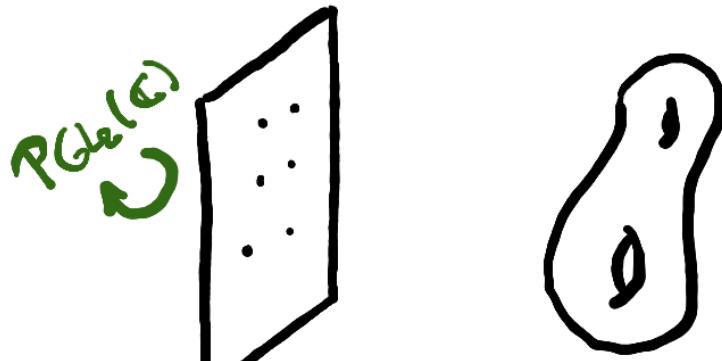
$$X \simeq \mathbb{C}/\Lambda = \langle 1, \tau \rangle$$

with  $\tau \in \mathfrak{H} = \{ z \in \mathbb{C} : \operatorname{Im}(z) > 0 \}$

$SL_2(\mathbb{Z}) \backslash \mathfrak{H}$

Genus 2: Any  $X_c$  is determined by

a)  $\mathbb{P}^1 \xrightarrow{2:1} X_c : y^2 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$



$$\text{Spec}(\mathbb{C}[\lambda_1, \lambda_2, \lambda_3]^G)$$

b) its period lattice

$$\text{Jac}(X) \cong \frac{\mathbb{C}^2}{\Lambda = \langle I_2, \Omega \rangle}$$

$$\text{with } \Omega \in \mathcal{H}_2 = \left\{ z \in M_{2 \times 2}(\mathbb{C}) : \begin{array}{l} z^T = z \\ \text{Im}(z) > 0 \end{array} \right\}$$

$M_2 = \mathbb{C}$ -points

of

$$\text{Spec} \left( \frac{\mathbb{Z}[\zeta_3][j_1, j_2, j_3, j_4]}{(j_1 j_3 - j_2^2 - 4 j_4)} \right)^{\mathbb{Z}_{\zeta_3}}$$

(Igusa).

$$Sp_4(\mathbb{Z}) \backslash \mathcal{H}_2$$

Bonus: Spaces  $M_{1,1}$  and  $M_2$  are interesting beyond their sets of  $\mathbb{C}$ -points!

E.g. They admit algebraic functions whose special values at CM points (singular moduli) are of tremendous arithmetic significance!



Want to classify also

$$X \rightarrow S$$

families of curves.

TODAY: Focus on two technical aspects

1. Representability (Deligne-Mumford stacks)

2. Compactification (Stable curves)

## § 1. Moduli spaces

Def: A contravariant functor

$$F : (\text{Sch}) \longrightarrow (\text{Set})$$

is representable if it is equiv to  $\text{Hom}(-, X)$   
for some scheme  $X$  ( $\simeq$  fine moduli scheme)

Note:  $X$  is uniquely determined by Yoneda

Many interesting examples exist!

Ex 1:  $F : S \longrightarrow \text{Iso classes } (E, p)$

where  $E \xrightarrow{\sim} S$  ell curve  
 $p$  point of order 11

$$X = \Gamma_1(11) \backslash \mathcal{H} : y^2 + y = x^3 - x$$

Generally, automorphisms form obstruction  
to representability

Ex 2:  $F: S \rightarrow$  Iso classes

$$C \rightarrow S$$

smooth proj curves genus g.

Suppose  $S = U_1 \cup U_2$  then can construct  
non-trivial  $C \rightarrow S$  by gluing

$$\left\{ \begin{array}{l} C_0 \times U_1 \rightarrow U_1 \\ C_0 \times U_2 \rightarrow U_2 \end{array} \right\} \text{ along } C_0 \times U_1 \Big|_{U_1 \cap U_2} \cong C_0 \times U_2 \Big|_{U_1 \cap U_2}$$

induced by non-trivial automorphism  
 $C_0 \xrightarrow{\sim} C_0$

If fine moduli scheme  $X$  existed, then  $C \rightarrow S$   
corresponds to

① Non-trivial map  $S \rightarrow X$

② ... which restricts to trivial

maps  $U_1, U_2 \rightarrow X$



Several solutions exist:

- ① Find best approximation to  $X$  in  $(\text{Sch})$  ( $=$  Coarse moduli scheme)
- ② Rigidity  $F$  to prevent the existence of automorphisms (e.g. pointed version)
- ③ Enlarge the category of schemes and force representability.

→ Stacks.

## § 1.1 Categories fibered in groupoids

Def: We say a category

$$\mathcal{A} \xrightarrow{\rho} (\text{Sch})$$

is fibered in groupoids if

a)  $\mathcal{A} \xrightarrow{\rho} (\text{Sch})$   $x \rightarrow y$

$\forall \exists$   $\rho$  (  $\dashrightarrow$  )  $U \xrightarrow{\varphi} V$

b)

$$x \xrightarrow{\quad} y \xleftarrow{\quad} z$$

$\forall \exists$   $\rho$  (  $\dashrightarrow$  )

$$U \xrightarrow{\quad} V \xleftarrow{\quad} W$$

This implies that

$$\mathcal{G}_U = \tilde{p}^*(U)$$

is a groupoid, i.e. every morphism is an isomorphism.

Def: The 2-category of categories fibered in groupoids over  $(\text{Sch})$  is defined by

OBJECTS: Categories fibered in groupoids

$$\mathcal{G} \xrightarrow{p} (\text{Sch})$$

1-MORPHISMS: Functors  $\mathcal{G}_1 \xrightarrow{F} \mathcal{G}_2$

$$\begin{array}{ccc} & F & \\ p_1 \swarrow & & \searrow p_2 \\ (\text{Sch}) & & \end{array}$$

2-MORPHISMS: Natural transformations

$$\eta: G_1 \rightarrow G_2 \quad \text{s.t.}$$

$$\begin{array}{ccc} \mathcal{G}_2 & \eta_x: G_1(x) \rightarrow G_2(x) & \\ p_2 \downarrow & \cdots & \\ (\text{Sch}) & U \xrightarrow{\cong} U & \end{array}$$

Note that every 2-morphism is an isomorphism!

To every scheme  $S$  we associate "representable" category

$$(\text{Sch}_S) \xrightarrow{\quad ? \quad} (\text{Sch})$$

$$p(x \rightarrow S) = x$$

$$p\left( \begin{array}{c} x_1 \rightarrow x_2 \\ \searrow \quad \swarrow \\ S \end{array} \right) = x_1 \rightarrow x_2.$$

and there is a canonical equivalence

$$\text{Hom}((\text{Sch}_S), \mathcal{S}) \cong \mathcal{J}_S$$

$\cong$

Now define  $M_g \xrightarrow{\quad ? \quad} (\text{Sch})$  by

OBJECTS:  $C \xrightarrow{\pi} S$  smooth proper  
geom. fibres  $C_{\bar{s}}$  proj. curves genus  $g$ .

MORPHISMS: Pullback diagrams

$$\begin{array}{ccc} C_1 & \xrightarrow{\quad} & C_2 \\ \downarrow \pi & & \downarrow \\ S_1 & \longrightarrow & S_2 \end{array}$$

$\dagger$  Functor  $F$  is tautologically representable by  $M_g$ , since

$$\text{Hom}((\text{Sch}), M_g) \cong (M_g)_S$$

$$= \{\pi: C \rightarrow S\}$$

... so what ?!

## § 1.2 Algebraic stacks

The category  $M_g$  ( $g \geq 2$ ) has rich structure

$\underline{\cong}$ :  $M_g$  is a stack, i.e.

① Curves defined locally on  $S$  + gluing isomorphisms  
→ Curve on  $S$

② Morphisms between curves loc on  $S$  + agreement  
→ Morphisms between curves on  $S$ .

B.  $M_g$  is Deligne-Mumford stack

① For all schemes  $X, Y \rightarrow M_g$ ,

fibre product

$X \times_{M_g} Y$  is representable.

② There exists a scheme  $X \rightarrow M_g$



s.t. for any other  $Y \rightarrow M_g$  the map

$X \times_{M_g} Y \longrightarrow Y$  surjective + étale.

Have already seen such  $X$  in examples

$$X \xrightarrow{\text{étale}} M_g \longrightarrow M_g$$

(coarse moduli scheme)

Since  $M_g$  is "close to a scheme" can make sense of many algebro-geom concepts.

Thm (Valuative criterion for properness)

$$f: \mathfrak{I}_1 \xrightarrow{\text{sep}} \mathfrak{I}_2$$

$$\text{Spec}(L) \rightarrow \text{Spec}(K) \rightarrow \mathfrak{I}_1 \quad f \text{ proper}$$

$$\downarrow \quad \quad \quad \downarrow \quad \dashrightarrow \quad \downarrow \quad \quad \quad \iff \quad \forall R \text{ dir}$$

$$\text{Spec}(R_1) \rightarrow \text{Spec}(R) \rightarrow \mathfrak{I}_2$$

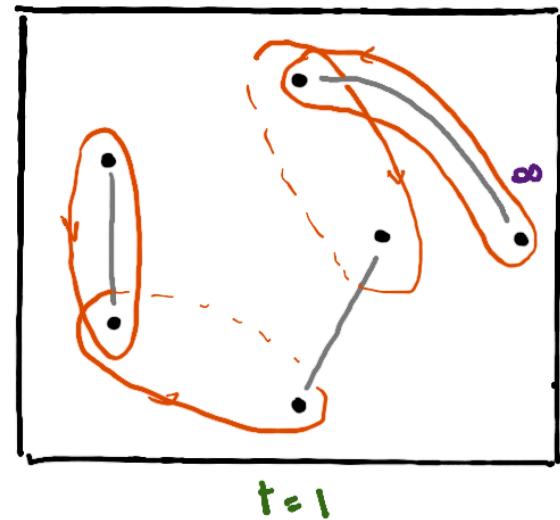
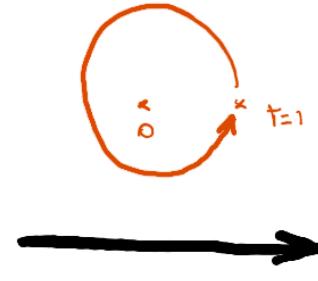
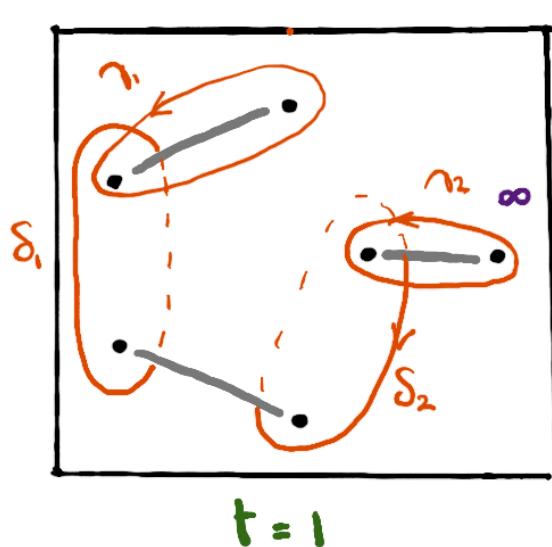
$\exists L/K$  finite

$K = \text{Frac}(R)$

## § 2. Properness

Is the moduli stack  $M_g$  proper?

Ex 1: Consider family  $y^2 = x^5 - t$



Basis  $\{\gamma_1, \gamma_2, \delta_1, \delta_2\}$  for homology

intersection matrix

$$\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

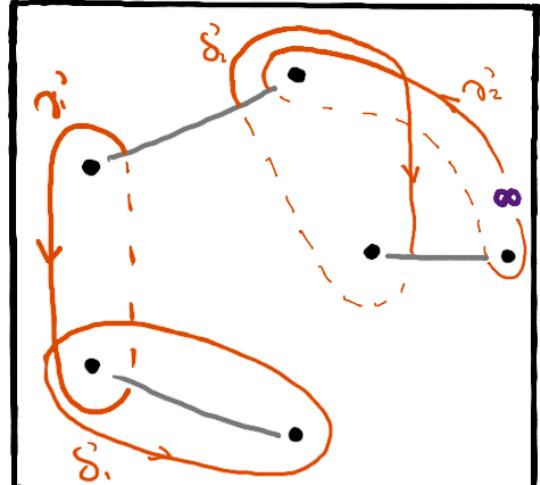
Action of loop on homology

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\delta_1 = \begin{pmatrix} -1 & -1 & 0 & 0 \end{pmatrix}$$

$$\delta_2 = \begin{pmatrix} 0 & -1 & -1 & 1 \end{pmatrix} =: M$$



$t=1$

Then  $M^{10} = \text{Id}_n$ !

Indeed, this family extends to smooth family  
of curves after finite base change

$$t^{10} \longleftrightarrow t$$

$$\text{Spec } \mathbb{C}[[t]] \rightarrow \text{Spec } \mathbb{C}[[t]] \rightarrow M_2 = \text{Spec } \mathbb{C}$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ \text{Spec } \mathbb{C}[t] & \xrightarrow{\quad \text{red dashed} \quad} & \text{Spec } (\mathbb{C}) \end{array}$$

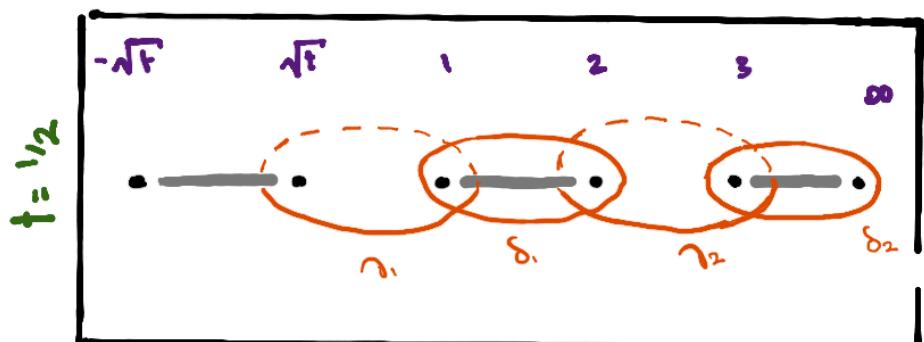
$$y^2 = x^5 - t^{10}$$

K

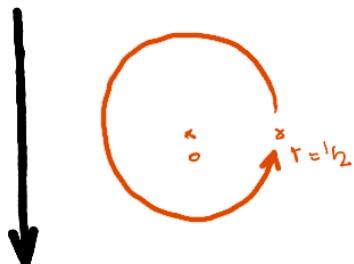
$$y^2 = x^5 - 1$$

Ex 2: Consider family.

$$y^2 = (x-1)(x-2)(x-3)(x^2-t)$$

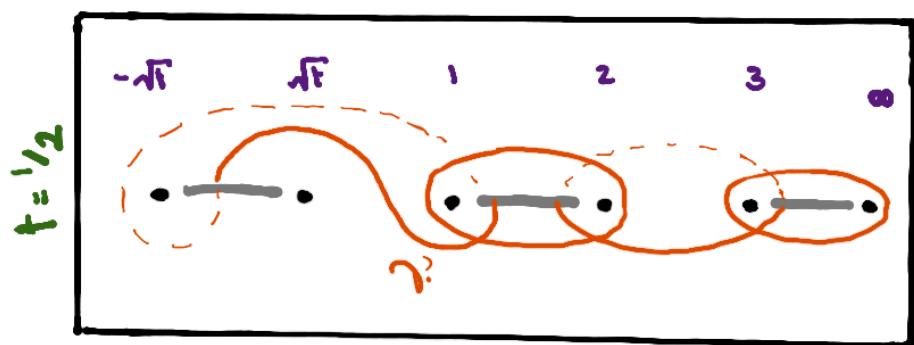


Basis for homology  
 $\{\gamma_1, \gamma_2, \delta_1, \delta_2\}$



Action of loop on homology

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =: M$$



M has infinite order!

Can never, after finite base change, extend to smooth family of curves above  $t=0$ .

$\Rightarrow M_g$  is Not proper

To compactify  $M_g \subset \overline{M}_g$ , use crucial def by Deligne-Mumford:

Def: A semi-stable curve of genus  $g$  is morphism

$$\pi : C \longrightarrow S \quad \text{proper flat}$$

whose geometric fibres  $C_s$  are reduced, non curves s.t

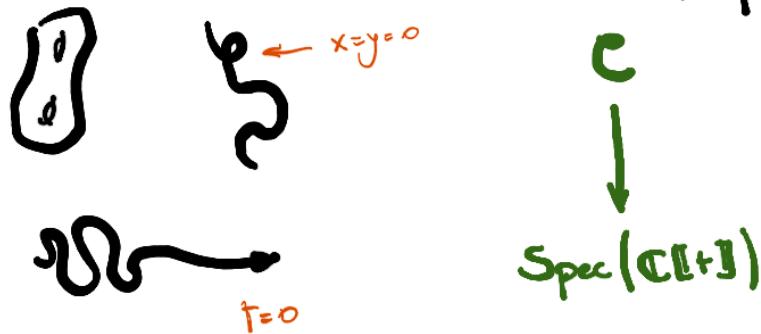
- ① only ordinary double points as singularities,  
i.e. étale locally

$$\text{Spec} \left( \overline{k}[[x,y]] / (xy) \right)$$

- ②  $\dim H^1(C_s, \mathcal{O}) = g$

It is stable if in addition every non-singular rational component meets other components in more than 2 pts.

Previous example is semi-stable over  $\mathbb{C}[[t]]$ , special fibre



Thm (Deligne-Mumford)

Suppose  $R$  DVR

$$K = \text{Frac}(R)$$

$C$  smooth, genus  $\geq 2$  curve over  $K$  of genus  $g \geq 2$

Then  $\exists$  finite  $L/K$  and stable curve

$$\tilde{C} \longrightarrow \text{Spec}(R_L) \quad \text{s.t. } \tilde{C}_L \simeq C \times L.$$

$\Downarrow$

Define  $\overline{\mathcal{M}}_g$  the moduli stack of stable curves of  $g \geq 2$

Then

- $\overline{\mathcal{M}}_g$  is a Deligne-Mumford stack
- $\overline{\mathcal{M}}_g$  is irreducible, proper, smooth over  $\text{Spec}(\mathbb{Z})$  of dimension  $3g-3$ .
- The boundary  $\overline{\mathcal{M}}_g - \mathcal{M}_g$  is a normal crossings divisor.