Toric Geometry

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The aim of this talk is to introduce the notion of *toric variety* and show how toric varieties admit a canonical log-scheme structure. The main reference for this talk is Chapter I of Fulton, "Introduction to Toric Varieties".

1 Affine Toric Varieties

Definition 1. *Let L be a lattice (a $\mathbb{Z}$-module isomoprhic to $\mathbb{Z}^n$ for some positive integer n), then a* ***convex polyhedral cone*** *is a subset of $L_\mathbb{R} = L \otimes_\mathbb{Z} \mathbb{R}$ of the form:*

$$\sigma = \{r_1 v_1 + \dots r_s v_s \mid r_i \geq 0\}$$

for some $v_1, \dots, v_s \in L_\mathbb{R}$.

The vectors $v_1, \dots, v_s$ are called generators for the cone σ. We define the dimension of a cone σ as the dimension of the $\mathbb{R}$ vector space generated by σ We define the dual of a cone σ as

$$\sigma^\vee := \{u \in L^*_\mathbb{R} \mid \langle u, v\rangle \geq 0 \text{ for all } v \in \sigma\}.$$

Moreover, we define a face $\tau \subseteq \sigma$ as the intersection of σ with any supporting hyperplane [1]

$$\tau = \sigma \cap u^\perp = \{v \in \sigma \mid \langle u, v\rangle = 0\}$$

for some $u \in \sigma^\vee$.

Example 1.

1. If we take $L = \mathbb{Z}^2$ and σ the cone generated by e_1 and e_2, then, $\sigma^\vee = \sigma$. Hence, the faces of σ are subsets of σ of the form $\sigma \cap u^\perp$ for $u \in \sigma$. For $u \in \sigma$ we have that $u^\perp \cap \sigma \neq \{0\}$ if and only if $u = e_1, e_2, 0$. Hence, as you might expect, the faces of σ are $\{\sigma, e_1, e_2, \{0\}\}$.
2. If we take $L = \mathbb{Z}^2$ and σ the cone generated by e_1 and $-e_1$, then $\sigma^\vee$ is generated by $e_2, -e_2$ and the only face of σ is σ itself.

We will list some fact about convex polyhedral cones without proving them

Fact 1.

1. *$(\sigma^\vee)^\vee = \sigma$;*
2. *Any face of a convex polyhedral cone is also a convex polyhedral cone;*
3. *Any intersection of faces is also a face;*
4. *Any face of a face is a face;*
5. *The dual of a convex polyhedral cone is a convex polyhedral cone (Farkas' Theorem).*

[1] A hyperplane such that σ is entairely cointained in one of the two closed half-space bounded by the hyperplane and σ has at least one bundary point on the hyperplane

Remark 1. The proof of all these properties uses the following result from the theory of convex sets:

($\star$) If σ is a convex polyhedral co ne and $v_0 \notin \sigma$, then there is some $u_0 \in \sigma^\vee$ with $\langle u_0, v_0 \rangle < 0$.

The proof of all these results can be found in Section 2, Chapter 1 of Fulton, "Introduction to Toric Varieties".

The first step, in order to associate to a convex polyhedral cone σ an affine variety, is to define a monoid attached to the cone, S_σ. We define

$$S_\sigma = \sigma^\vee \cap M = \{u \in M \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}.$$

1. $\sigma \in \mathbb{R}^2$ be the cone generated by $\{e_2, 2e_1 - e_2\}$. Then,

   $$S_\sigma = \{\alpha e_1^* + \beta e_2^* \in M \mid (\alpha e_1^* + \beta e_2^*)(e_2) \geq 0 \text{ and } (\alpha e_1^* + \beta e_2^*)(2e_1 - e_2) \geq 0\}.$$

   Hence
   $$S_\sigma = \{\alpha e_1^* + \beta e_2^* \in M \mid \beta \geq 0 \text{ and } 2\alpha \geq \beta\} = \{\alpha e_1^* + \beta e_2^* \in M \mid 2\alpha \geq \beta \geq 0\}$$
   which is generated by $\{e_1^*, e_1^* + e_2^*, e_1^* + 2e_2^*\}$.

2. $\sigma \in \mathbb{R}^2$ be the cone generated by $\{e_2, \lambda e_1 - e_2\}$, with λ positive irrational number. Then,

   $$S_\sigma = \{\alpha e_1^* + \beta e_2^* \in M \mid \lambda\alpha \geq \beta \geq 0\}.$$

   Suppose by contradiction that S_σ is finitely generated, generated by $\{(\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)\}$. Using the irrationality of λ we get that there exist α, β such that $\alpha e_1^* + \beta e_2^* \in S_\sigma$ and

   $$\max_j \left\{ \frac{\beta_j}{\alpha_j} \right\} < \frac{\beta}{\alpha} < \lambda.$$

   There exists some natural numbers $n_1, \dots, n_k$ such that

   $$\frac{\beta}{\alpha} = \frac{n_1\beta_1 + \cdots + n_k\beta_k}{n_1\alpha_1 + \cdots + n_k\alpha_k}$$

   let $i_0 := \max_{j=1,\dots,k}\{n_j\beta_k\}$, then

   $$\frac{\beta}{\alpha} \leq \frac{n_{i_0}\beta_{i_0}}{n_{i_0}\alpha_{i_0}} = \frac{\beta_{i_0}}{\alpha_{i_0}} \leq \max_j \left\{ \frac{\beta_j}{\alpha_j} \right\}$$

   which give us the desired contradiction.

3. More in general, if σ is the cone generated by e_2 and $\lambda e_1 - e_2$, with λ positive, then S_σ is finitely generated if and only if λ is rational if and only if σ admits a set of generators from $\mathbb{Z}^2$. Indeed, if $\lambda = m/n$ is rational, then
   $$\sigma = \langle e_2, me_1 - ne_2 \rangle$$
   and $\sigma^\vee$ is generated by $e_1^*, e_1^* + e_2^*, \dots, e_1^* + me_2^*$.

Definition 2. *A covex polyhedral cone is said to be* ***rational*** *if its generators can be taken from L.*

It can be proven that if σ is rational then also its dual $\sigma^\vee$ is rational. Using the latter result it is possible to prove the following:

Lemma 1 (Gordon's Lemma)**.** *If σ is a rational convex polyhedral cone, then $S_\sigma := \sigma^\vee \cap M$ is a finitely generated semigroup.*

Proof. See Proposition 1, Chapter 1 of Fulton, "Introduction to Toric Varieties". □

Example 2. Let $L = \mathbb{Z}^2 \subseteq \mathbb{R}^2$ with canonical basis e_1, e_2 and let e_1^*, e_2^* be the canonical basis of $M = L^*$.

Definition 3. *A cone is called* ***strongly convex*** *if it contains no nonzero linear subspace.*

Remark 2. It can be proven that being strongly convex is equivalent to ask that there is a vector $u \in \sigma^\vee$ such that $\sigma \cap u^\perp = \{0\}$, namely $\{0\}$ is a face of σ.

Any additive semigroup determines a "group ring" $\mathbb{C}[S]$, which is a commutative $\mathbb{C}$ algebra. As a $\mathbb{C}$ vector space it has a basis $\{X^u\}_{u\in S}$, and the multiplication is determined by $X^uX^v = X^{u+v}$'. Clearly generators of the semigroup S determine generators for the $\mathbb{C}$ algebra $\mathbb{C}[S]$.

Definition 4. *We associated to a strongy convex rational cone the finitely generated $\mathbb{C}$ scheme*

$$U_\sigma := Spec(\mathbb{C}[S_\sigma]).$$

U_σ is called ***affine toric varity****.*

We have the following key proposition

Proposition 2. *If $\tau \subseteq \sigma$ is a face of σ, then $U_\tau \to U_\sigma$ embeds U_τ as a principal open subset of U_σ.*

Proof. First one should prove the following: if $\tau = \sigma \cap u^\perp$ then we can assume that $u \in \sigma^\vee \cap M$, then

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0} \cdot (-u).$$

Therefore, every element of the basis of $\mathbb{C}[S_\tau]$ can be written in the form X^{w-pu} with $w \in S_\sigma$. Hence, $\mathbb{C}[S_\tau] = (\mathbb{C}[S_\sigma])_{\{1,1/X^u,\dots\}}$.

□

Example 3.

1. Let $L = \mathbb{Z}^n$ with canonical basis $\{e_1, \dots, e_n\}$ and let σ be the cone with generators $e_1, \dots, e_k$ for some $k \leq n$. Then S_σ is generated by $\{e_1^*, \dots, e_k^*, \pm e_{k+1}^*, \dots \pm e_n^*\}$. Hence,

   $$\mathbb{C}[S_\sigma] = \mathbb{C}[X_1, \dots, X_k, X_{k+1}, X_{k+1}^{-1}, \dots, X_n, X_n^{-1}]$$

   and U_σ is a product (fibred product) of the affine k space with an $(n-k)$ dimensional torus.

2. Back to Example 2.1: σ is generated by e_2 and $2e_1 - e_2$. Hence $\sigma^\vee$ is generated by $e_1^*, e_1^* + e_2^*, e_1^* + 2e_2^*$.

   $$\mathbb{C}[S_\sigma] = \mathbb{C}[X_1, X_1X_2, X_1X_2^2] = \mathbb{C}[U,V,W]/(V^2 - UW)$$

3. A singular example: let σ be the cone generated but $e_1, e_2, e_3, e_1 + e_3 - e_2$. Then S_σ is generated by $\{e_1^*, e_3^*, e_1^* + e_2^*, e_2^* + e_3^*\}$; hence

   $$\mathbb{C}[S_\sigma] = \mathbb{C}[X_1, X_3, X_2X_1, X_2X_3] = \mathbb{C}[U,V,W,T]/(UT - VW)$$

Remark 3. All of the semigroups of the form S_σ for some strongly convex rational polyhedral cone σ are sub-semigroups of the group $M = S_0$. As a semigroup, M has generators $\pm e_1^*, \dots, \pm e_n^*$ so $\mathbb{C}[M] = \mathbb{C}[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$. Hence $C[S_\sigma]$ is a subalgebra of $\mathbb{C}[M]$ and in particular it is a domain.

In particular, by the previous proposition, we get that every toric affine variety contains $U_{\{0\}}$ as a principal open subset and

$$U_{\{0\}} = \mathrm{Spec}(\mathbb{C}[M]) = \mathrm{Spec}(\mathbb{C}[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}])$$

is the n dimensional torus!

Fact 2. *It can be proven, using the properties of a strongly convex rational cone σ, that every affine toric variety is* ***normal****. Furthermore, the cone σ is* ***regular*** *(i.e. it admits a system of generators that can be completed to a basis of the lattice L) if and only if the associated toric variety is regular.*

2 Fans and toric varieties

Definition 5. *By a **fan** Δ in L is meant a finite set of rational strongly convex polyhedral cones σ in $L_{\mathbb{R}}$ such that*

1. *Each face of a cone in Δ is also a cone in Δ;*
2. *The intersection of two cones in Δ is a face of each.*

From a fan Δ the toric variety $X(\Delta)$ is constructed by taking the disjoint union of the affine toric varieties U_σ, and gluing as follows: for cones σ, τ the intersection is a face of each, so $U_{\sigma\cap\tau}$ is identified as a principal open subvariety of both; glue U_σ and U_τ by this identification on this open subvarieties. Note that, these identifications are compatible, using the order preserving nature of the correspondence from cones to affine varieties.

Example 4.

1. Take $L = \mathbb{Z}$ and $\Delta = \{\sigma_+ := \mathbb{R}_{\geq 0}, \sigma_- := \mathbb{R}_{\leq 0}, \{0\}\}$. Then, $U_+ = \mathrm{Spec}(\mathbb{C}[X])$, $U_- = \mathrm{Spec}(\mathbb{C}[Y])$ and the gluing on the overlap is given by:
$$\begin{aligned} U_0 = \mathrm{Spec}(\mathbb{C}[X, X^{-1}]) &\to U_0 = \mathrm{Spec}(\mathbb{C}[Y, Y^{-1}]) \\ X &\mapsto Y^{-1} \end{aligned}$$
Hence, $X(\Delta) = \mathbb{P}^1_{\mathbb{C}}$.

2. $n = 2$ and Δ the fan "generated" by $\sigma_0 = \{e_1, e_1 + e_2\}$ and $\sigma_1 = \{e_2, e_1 + e_2\}$. Then
$$U_{\sigma_0} = \mathrm{Spec}(\mathbb{C}[X, X^{-1}Y])$$
and
$$U_{\sigma_1} = \mathrm{Spec}(\mathbb{C}[Y, XY^{-1}]).$$
On the intersection we glue through the isomorphism sending $X^{-1}Y$ to $(XY^{-1})^{-1}$. Namely, the resulting variety is a blow-up of the affine plane on the origin.

3 The log structure

We start this section with a brief recall of what Pim has introduced during his talk.

A pre log structure on a scheme X is a sheaf of monoids M on the topological space X together with a morphism of monoids $\alpha : M \to \mathcal{O}_X$. Moreover, we define the log structure $M^a \to \mathcal{O}_X$ associated to it as the pushout of the following square

$$\begin{tikzcd} \alpha^{-1}\mathcal{O}_X^\times \arrow[r] \arrow[d, "\alpha"] & M \arrow[d] \\ \mathcal{O}_X^\times \arrow[r] & M^a \end{tikzcd}$$

Finally a chart of X is a strict morphism $X \to (P \to \mathbb{Z}[P])$ and a log scheme X is fs if étale locally it has a chart modelled over an fs monoid.

On an affine toric variety we have a natural pre-log structure given by $S_\sigma \hookrightarrow \mathbb{C}[S_\sigma]$. Hence, we can put on U_σ the associated log structure. A chart on U_σ is given by the map on log scheme induced by the following map on pre log schemes
$$(S_\sigma, U_\sigma) \to (S_\sigma, \mathrm{Spec}(\mathbb{Z}[S_\sigma])).$$

Proposition 3. *S_σ is fine and saturated.*

Proof. Saturatedness follows almost immediately from the definition of S_σ.

We recall that by fine we mean finitely generated and integral. We have already mentioned the fact σ being rational implies S_σ finitely generated. We have in some sense already proven also the integrality of S_σ. Indeed, σ being strongly convex is equivalent to $\{0\}$ being a face of σ, which implies that S_σ is a submonoid of $S_{\{0\}} = \mathbb{Z}^n$, which is of course integral. □

Hence an affine toric variety admits a fs log structure.

In a similar way one can show that every toric variety $X(\Delta)$ admits a fs log structure.

4 Extra

As you may expect many properties of a toric variety can be deduced from the fan associated to it. We will mention some of them.

Definition 6. *We say that a cone is* ***regular*** *if it admits a system that can be completed to a basis of the lattice N. A fan is regular if every cone in the fan is regular.*

Proposition 4. *A fan Δ is regular if and only if the associated toric variety $X(\Delta)$ is smooth.*

Definition 7. *A fan is* ***complete*** *if its cones covers $\mathbb{R}^n$, i.e. $|\Delta| = \mathbb{R}^n$.*

Proposition 5. *A fan Δ is complete if and only if the associated toric variety $X(\Delta)$ is compact.*

Proposition 6. *Toric varieties are normal (i.e. integrally closed) and separated.*

Finally it can be proven that it is possible to give the following definition of toric variety:

Theorem 7. *A toric variety is an algebraic normal variety X that contains a torus T as a dense open subset, together with an action $T \times X \to X$ that extends the natural action of the torus T on itself.*

A proof of all these results can be found in Section 3.4 of Jean-Paul Brasselet, "Introduction to Toric Varieties".