Left-multiplication by J yields

$$J\mathbf{e}_5 = \lambda\mathbf{e}_5 + \mathbf{e}_4, \qquad J\mathbf{e}_4 = \lambda\mathbf{e}_4 + \mathbf{e}_3, \qquad J\mathbf{e}_3 = \lambda\mathbf{e}_3 + \mathbf{e}_2,$$
$$J\mathbf{e}_2 = \lambda\mathbf{e}_2 + \mathbf{e}_1, \qquad J\mathbf{e}_1 = \mathbf{0}.$$

The matrix J in Example 2 is an example of a *Jordan block* matrix.

DEFINITION 9.6 Jordan Block

An $m \times m$ matrix is a **Jordan block** if it is structured as follows:

1. All diagonal entries are equal.
2. Each entry immediately above a diagonal entry is 1.
3. All other entries are zero.

Thus, the matrix J in Example 2 is a Jordan block. However, the matrix in Example 1 is not a Jordan block, since the entry 5 at the bottom of the diagonal does not have a 1 just above it. A Jordan block has the properties described in the next theorem. These properties were illustrated in Example 2, and we leave a formal proof to you if you desire one. Notice that, for an $m \times m$ Jordan block

$$J = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix},$$

we have just one string:

$$J - \lambda I: \qquad \mathbf{e}_m \to \mathbf{e}_{m-1} \to \cdots \to \mathbf{e}_2 \to \mathbf{e}_1 \to \mathbf{0}.$$

THEOREM 9.8 Properties of a Jordan Block

Let J be an $m \times m$ Jordan block with diagonal entries all equal to λ. Then the following properties hold:

1. $(J - \lambda I)\mathbf{e}_i = \mathbf{e}_{i-1}$ for $1 < i \le m$, and $(J - \lambda I)\mathbf{e}_1 = \mathbf{0}$.
2. $(J - \lambda I)^m = O$, but $(J - \lambda I)^i \ne O$ for $i < m$.
3. $J\mathbf{e}_i = \lambda\mathbf{e}_i + \mathbf{e}_{i-1}$ for $1 < i \le m$, whereas $J\mathbf{e}_1 = \lambda\mathbf{e}_1$.

Jordan Canonical Forms

We have seen that not every $n \times n$ matrix is diagonalizable. It is our purpose in this section to show that every $n \times n$ matrix is similar to a matrix having all entries 0 except for those on the diagonal and entries 1 immediately above some diagonal entries; each 1 above a diagonal entry must have the same number on its left as below it on the diagonal. An example of such a matrix is

$$J = \begin{bmatrix} -i & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix}. \tag{4}$$

As the shading indicates, this matrix J is comprised of four Jordan blocks, placed corner-to-corner along the diagonal.

DEFINITION 9.7 Jordan Canonical Form

An $n \times n$ matrix J is a **Jordan canonical form** if it consists of Jordan blocks, placed corner-to-corner along the main diagonal, as in matrix (4), with only zero entries outside these Jordan blocks.

Every diagonal matrix is a Jordan canonical form, because each diagonal entry can be viewed as being the sole entry in a 1×1 Jordan block. Notice that matrix (4) contains the 1×1 Jordan block [2]. Notice, too, that the breaks between the Jordan blocks in matrix (4) occur where some diagonal entry has a 0 rather than 1 immediately above it.

EXAMPLE 3 Is the matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 7 \end{bmatrix}$$

a Jordan canonical form? Why?

SOLUTION This matrix is not a Jordan canonical form. Because not all diagonal entries are equal, there should be at least two Jordan blocks present in order for the matrix to be a Jordan canonical form, and [7] should be a 1×1 Jordan block. However, the entry immediately above 7 is not 0. Consequently, this matrix is not a Jordan canonical form. ■

EXAMPLE 4 Describe the effect of matrix J in Eq. (4) on each of the standard basis vectors in $\mathbb{C}^8$. Then give the eigenvalues and eigenspaces of J. Finally, find the dimension of the nullspace of $(J - \lambda I)^k$ for each eigenvalue λ of J and for each positive integer k.

SOLUTION We find that

$$\begin{aligned} J\mathbf{e}_3 &= -i\mathbf{e}_3 + \mathbf{e}_2, & J\mathbf{e}_2 &= -i\mathbf{e}_2 + \mathbf{e}_1, & J\mathbf{e}_1 &= -i\mathbf{e}_1, \\ J\mathbf{e}_5 &= -i\mathbf{e}_5 + \mathbf{e}_4, & J\mathbf{e}_4 &= -i\mathbf{e}_4, \\ J\mathbf{e}_6 &= 2\mathbf{e}_6, \\ J\mathbf{e}_8 &= 5\mathbf{e}_8 + \mathbf{e}_7, & J\mathbf{e}_7 &= 5\mathbf{e}_7. \end{aligned}$$

The eigenvalues of J are $-i$, 2, and 5, which have algebraic multiplicities of 5, 1, and 2, respectively. The eigenspaces of J are $E_{-i} = \mathrm{sp}(\mathbf{e}_1, \mathbf{e}_4)$, $E_2 = \mathrm{sp}(\mathbf{e}_6)$, and $E_5 = \mathrm{sp}(\mathbf{e}_7)$, as you can easily check.

The effect of $J - (-i)I$ on the first five standard basis vectors is given by the two strings

$$J + iI: \qquad \begin{aligned} \mathbf{e}_3 \to \mathbf{e}_2 \to \mathbf{e}_1 &\to \mathbf{0}, \\ \mathbf{e}_5 \to \mathbf{e}_4 &\to \mathbf{0}. \end{aligned} \tag{5}$$

The 3×3 lower right-hand corner of $J + iI$ describes the action of $J + iI$ on $\mathbf{e}_6$, $\mathbf{e}_7$, and $\mathbf{e}_8$. Because this 3×3 matrix has a nonzero determinant, it causes $J + iI$ to carry these three vectors into three independent vectors, and the same is true of all powers of $J + iI$. Thus we can determine the dimension of the nullspace of $J + iI$ by diagram (5), and we find that

$J + iI$ has nullspace $\mathrm{sp}(\mathbf{e}_1, \mathbf{e}_4)$ of dimension 2,
$(J + iI)^2$ has nullspace $\mathrm{sp}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4, \mathbf{e}_5)$ of dimension 4,
$(J + iI)^3$ has nullspace $\mathrm{sp}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5)$ of dimension 5,
$(J + iI)^k$ has the same nullspace as that of $(J + iI)^3$ for $k > 3$.

By a similar argument, we find that

$(J - 2I)^k$ has nullspace $\mathrm{sp}(\mathbf{e}_6)$ of dimension 1 for $k \geq 1$,
$J - 5I$ has nullspace $\mathrm{sp}(\mathbf{e}_7)$ of dimension 1,
$(J - 5I)^k$ has nullspace $\mathrm{sp}(\mathbf{e}_7, \mathbf{e}_8)$ of dimension 2 for $k > 1$. ■

HISTORICAL NOTE THE JORDAN CANONICAL FORM appears in the *Treatise on Substitutions and Algebraic Equations*, the chief work of the French algebraist Camille Jordan (1838–1921). This text, which appeared in 1870, incorporated the author's group-theory work over the preceding decade and became the bible of the field for the remainder of the nineteenth century. The theorem containing the canonical form actually deals not with matrices over the real numbers, but with matrices with entries from the finite field of order p. And as the title of the book indicates, Jordan was not considering matrices as such, but the linear substitutions that they represented.

Camille Jordan, a brilliant student, entered the Ecole Polytechnique in Paris at the age of 17 and practiced engineering from the time of his graduation until 1885. He thus had ample time for mathematical research. From 1873 until 1912, he taught at both the Ecole Polytechnique and the Collège de France. Besides doing seminal work on group theory, he is known for important discoveries in modern analysis and topology.

EXAMPLE 5 Suppose a 9×9 Jordan canonical form J has the following properties:

1. $(J - 3iI)^k$ has rank 7 for $k = 1$, rank 5 for $k = 2$, and rank 4 for $k \geq 3$,
2. $(J + I)^j$ has rank 6 for $j = 1$ and rank 5 for $j \geq 2$.

Find the Jordan blocks that appear in J.

SOLUTION Because the rank of $J - 3iI$ is 7, the dimension of its nullspace is $9 - 7 = 2$, so $3i$ is an eigenvalue of geometric multiplicity 2. It must give rise to two Jordan blocks. In addition, $J - 3iI$ must annihilate two eigenvectors $\mathbf{e}_r$ and $\mathbf{e}_s$ in the standard basis. Because the rank of $(J - 3iI)^2$ is 5, its nullspace must have dimension 4, so in a diagram of the effect of $J - 3iI$ on the standard basis, we must have $(J - 3iI)\mathbf{e}_{r+1} = \mathbf{e}_r$ and $(J - 3iI)\mathbf{e}_{s+1} = \mathbf{e}_s$. Because $(J - 3iI)^k$ has rank 4 for $k \geq 3$, its nullity is 5, and we have just one more standard basis vector—either $\mathbf{e}_{r+2}$ or $\mathbf{e}_{s+2}$—that is annihilated by $(J - 3iI)^3$. Thus, the two Jordan blocks in J that have $3i$ on the diagonal are

$$J_1 = \begin{bmatrix} 3i & 1 & 0 \\ 0 & 3i & 1 \\ 0 & 0 & 3i \end{bmatrix} \quad \text{and} \quad J_2 = \begin{bmatrix} 3i & 1 \\ 0 & 3i \end{bmatrix}.$$

Because $J + 1I$ has rank 6, its nullspace has dimension $9 - 6 = 3$, so -1 is an eigenvalue of geometric multiplicity 3 and gives rise to three Jordan blocks. Because $(J + I)^j$ has rank 5 for $j \geq 2$, its nullspace has dimension 4, so $(J + I)^2$ annihilates a total of four standard basis vectors. Thus, just one of these Jordan blocks is 2×2, and the other two are 1×1. The Jordan blocks arising from the eigenvalue -1 are then

$$J_3 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad J_4 = J_5 = [-1].$$

The matrix J might have these blocks in any order down its diagonal. Symbolically, we might have

$$J = \begin{bmatrix} J_3 & & & & \\ & J_1 & & \Large 0 & \\ & & J_4 & & \\ & \Large 0 & & J_2 & \\ & & & & J_5 \end{bmatrix}, \quad J = \begin{bmatrix} J_4 & & & & \\ & J_5 & & \Large 0 & \\ & & J_2 & & \\ & \Large 0 & & J_1 & \\ & & & & J_3 \end{bmatrix},$$

or any other order. ■

Jordan Bases

If an $n \times n$ matrix A is similar to a Jordan canonical form J, we call J a **Jordan canonical form of** A. When this is the case, there exists an invertible matrix C such that $C^{-1}AC = J$. We know that similar matrices represent the same linear transformation, but with respect to different bases. Thus, if A is similar to J, there must exist a basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ of $\mathbb{C}^n$ with the same schematic string properties relative to A that the standard ordered basis has relative to the matrix J. We proceed to define such a *Jordan basis*.

DEFINITION 9.8 Jordan Basis

Let A be an $n \times n$ matrix. An ordered basis $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ of $\mathbb{C}^n$ is a **Jordan basis for** A if, for $1 \leq j \leq n$, we have either $A\mathbf{b}_j = \lambda\mathbf{b}_j$ or $A\mathbf{b}_j = \lambda\mathbf{b}_j + \mathbf{b}_{j-1}$, where λ is an eigenvalue of A that we say is **associated with** $\mathbf{b}_j$. If $A\mathbf{b}_j = \lambda\mathbf{b}_j + \mathbf{b}_{j-1}$, we require that the eigenvalue associated with $\mathbf{b}_{j-1}$ also be λ.

If an $n \times n$ matrix A has a Jordan basis B, then the matrix representation of the linear transformation $T(\mathbf{z}) = A\mathbf{z}$ relative to B must be a Jordan canonical form. We know then that $J = C^{-1}AC$, where C is the $n \times n$ matrix whose jth column vector is the jth vector $\mathbf{b}_j$ in B. In a moment we will prove that, for every square matrix, there is an associated Jordan basis, and consequently that every square matrix is similar to a Jordan canonical form. First, though, we outline a method for the computation of a Jordan canonical form of A.

Finding a Jordan Canonical Form of A

1. Find the eigenvalues of A.
2. For each eigenvalue λ, compute the rank of $(A - \lambda I)^k$ for consecutive values of k, starting with $k = 1$, until the same rank is obtained for two consecutive values of k.
3. From the data generated, find a Jordan canonical form for A, as in Example 5.

We now illustrate this technique.

EXAMPLE 6 Find a Jordan canonical form of the matrix

$$A = \begin{bmatrix} 2 & 5 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

SOLUTION Because A is an upper-triangular matrix, we see that the eigenvalues of A are $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = \lambda_4 = \lambda_5 = -1$. Now

$$A - \lambda_1 I = A - 2I = \begin{bmatrix} 0 & 5 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}$$

has rank 4 and consequently has a nullspace of dimension 1. We find that

$$(A - 2I)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 9 \end{bmatrix},$$

which has rank 3 and therefore has a nullspace of dimension 2. Furthermore,

$$(A - 2I)^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -27 & 0 & 0 \\ 0 & 0 & 0 & -27 & 0 \\ 0 & 0 & 0 & 0 & -27 \end{bmatrix}$$

has the same rank and nullity as $(A - 2I)^2$. Thus we have $A\mathbf{b}_1 = 2\mathbf{b}_1$ and $A\mathbf{b}_2 = 2\mathbf{b}_2 + \mathbf{b}_1$ for some Jordan basis $B = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5)$ for A. There is just one Jordan block associated with $\lambda_1 = 2$—namely,

$$J_1 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

For the eigenvalue $\lambda_3 = -1$, we find that

$$A - \lambda_3 I = A + I = \begin{bmatrix} 3 & 5 & 0 & 0 & 1 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which has rank 2 and therefore has a nullspace of dimension 3. Because -1 is an eigenvalue of both algebraic multiplicity and geometric multiplicity 3, we realize that $J_2 = J_3 = J_4 = [-1]$ are the remaining Jordan blocks. This is confirmed by the fact that

$$(A + I)^2 = \begin{bmatrix} 9 & 30 & 0 & 0 & 3 \\ 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

again has rank 2 and nullity 3. Thus, a Jordan canonical form for A is

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

■

EXAMPLE 7 Find a Jordan basis for matrix A in Example 6.

SOLUTION For the part of a Jordan basis associated with the eigenvalue 2, we need to find a vector $\mathbf{b}_2$ in the nullspace of $(A - 2I)^2$ that is not in the nullspace of $A - 2I$; then we may take $\mathbf{b}_1 = (A - 2I)\mathbf{b}_2$. From the computation of $A - 2I$ and $(A - 2I)^2$ in Example 6, we see that we can take

$$\mathbf{b}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \text{and then} \quad \mathbf{b}_1 = (A - 2I)\mathbf{b}_2 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

For $\mathbf{b}_3$, $\mathbf{b}_4$, and $\mathbf{b}_5$, we need only take a basis for the nullspace of $A + I$. We see that we can take

$$\mathbf{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{b}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{b}_5 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 3 \end{bmatrix}.$$

■

In Example 7, it was easy to find vectors in a Jordan basis corresponding to the eigenvalue 2 whose geometric multiplicity is less than its algebraic multiplicity, because only one Jordan block corresponds to the eigenvalue 2. We now indicate how a Jordan basis can be constructed when more than one such block corresponds to a single eigenvalue λ. Let N_r be the nullspace of $(A - \lambda I)^r$ for $r \geq 1$, and suppose (for example) that $\dim(N_1) = 4$, $\dim(N_2) = 7$, and $\dim(N_r) = 8$ for $r \geq 3$. Then a Jordan basis for A contains four strings corresponding to λ, which we may represent as

$$\begin{aligned} \mathbf{b}_3 \to \mathbf{b}_2 \to \mathbf{b}_1 &\to \mathbf{0}, \\ \mathbf{b}_5 \to \mathbf{b}_4 &\to \mathbf{0}, \\ \mathbf{b}_7 \to \mathbf{b}_6 &\to \mathbf{0}, \\ \mathbf{b}_8 &\to \mathbf{0}. \end{aligned}$$

To find the first and longest of these strings, we *compute* a basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_8\}$ for the nullspace N_3 of $(A - \lambda I)^3$. The preceding strings show that multiplication of all of the vectors in N_3 on the left by $(A - \lambda I)^2$ yields a space of dimension 1, so at least one of the vectors $\mathbf{v}_i$ has the property that $(A - \lambda I)^2\mathbf{v}_i \neq \mathbf{0}$.

Let $\mathbf{b}_3$ be such a vector, and set $\mathbf{b}_2 = (A - \lambda I)\mathbf{b}_3$ and $\mathbf{b}_1 = (A - \lambda I)\mathbf{b}_2$. It is not difficult to show that $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ must be independent. Thus we have found the first string.

Now $\mathbf{b}_1$ and $\mathbf{b}_2$ lie in N_2, and we can expand the independent set $\{\mathbf{b}_1, \mathbf{b}_2\}$ to a basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{w}_1, \ldots, \mathbf{w}_5\}$ of N_2. Again, the strings displayed earlier show that multiplication of the vectors in N_2 on the left by $A - \lambda I$ must yield a space of dimension 3, so there exist two vectors $\mathbf{w}_i$ and $\mathbf{w}_j$ such that the vectors $\mathbf{b}_1$, $(A - \lambda I)\mathbf{w}_i$, and $(A - \lambda I)\mathbf{w}_j$ are independent. Let $\mathbf{b}_5 = \mathbf{w}_i$ and $\mathbf{b}_4 = (A - \lambda I)\mathbf{b}_5$, while $\mathbf{b}_7 = \mathbf{w}_j$ and $\mathbf{b}_6 = (A - \lambda I)\mathbf{b}_7$. It can be shown that the vectors $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_7$ are independent. Finally, we expand the set $\{\mathbf{b}_1, \mathbf{b}_4, \mathbf{b}_6\}$ to a basis $\{\mathbf{b}_1, \mathbf{b}_4, \mathbf{b}_6, \mathbf{b}_8\}$ for N_1 to complete the portion of the Jordan basis corresponding to λ.

Although we know the techniques for finding bases for the nullspaces N_i and for expanding a given set of independent vectors to a basis, significant pencil-and-paper illustrations of this construction would be cumbersome, so we do not include them here. Any Jordan bases requested in the exercises can be found as in Example 7.

An application of the Jordan canonical form to differential equations is indicated in Exercise 32. We mention that computer-aided computation of a Jordan canonical form for a square matrix is not a stable process. Consider, for example, the matrix

$$A = \begin{bmatrix} 2 & c \\ 0 & 2 \end{bmatrix}.$$

If $c = 10^{-100}$, then the Jordan canonical form of A has 1 as its entry in the upper right-hand corner; but if $c = 0$, that entry is 0.

Existence of a Jordan Form for a Square Matrix

To demonstrate the existence of a Jordan canonical form similar to an $n \times n$ matrix A, we need only show that we have a Jordan basis B for A. Let us formalize the concept of a *string* in a Jordan basis $B = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$. Let λ be an eigenvalue of A. If $A\mathbf{b}_i = \lambda\mathbf{b}_i$ and $A\mathbf{b}_k = \lambda\mathbf{b}_k + \mathbf{b}_{k-1}$ for $i < k < j$, while $A\mathbf{b}_j \neq \lambda\mathbf{b}_j + \mathbf{b}_{j-1}$, we refer to the sequence $\mathbf{b}_i, \mathbf{b}_{i+1}, \ldots, \mathbf{b}_{j-1}$ as a **string** of basis vectors **starting** at $\mathbf{b}_{j-1}$, **ending** at $\mathbf{b}_i$, and **associated with** λ. This string is represented by the diagram

$$A - \lambda I: \qquad \mathbf{b}_{j-1} \to \cdots \to \mathbf{b}_{i+1} \to \mathbf{b}_i \to \mathbf{0}.$$

THEOREM 9.9 Jordan Canonical Form of a Square Matrix

Let A be a square matrix. There exists an invertible matrix C such that the matrix $J = C^{-1}AC$ is a Jordan canonical form. This Jordan canonical form is unique, except for the order of the Jordan blocks of which it is composed.

PROOF We use a proof due to Filippov. First we note that it suffices to prove the theorem for matrices A having 0 as an eigenvalue. Observe that, if λ is an eigenvalue of A, then 0 is an eigenvalue of $A - \lambda I$. Now if we can find C such that $C^{-1}(A - \lambda I)C = J$ is a Jordan canonical form, then $C^{-1}AC = J + \lambda I$ is also a Jordan canonical form. Thus, we restrict ourselves to the case where A has an eigenvalue of 0.

In order to find a Jordan canonical form for A, it is useful to consider also the linear transformation $T: \mathbb{C}^n \to \mathbb{C}^n$, where $T(\mathbf{z}) = A\mathbf{z}$; a Jordan basis for A is considered to be a Jordan basis for T. We will prove the existence of a Jordan basis for any such linear transformation by induction on the dimension of the domain of the transformation.

If T is a linear transformation of a one-dimensional vector space $\text{sp}(\mathbf{z})$, then $T(\mathbf{z}) = \lambda\mathbf{z}$ for some $\lambda \in \mathbb{C}$, and $\{\mathbf{z}\}$ is the required Jordan basis. (The matrix of T with respect to this ordered basis is the 1×1 matrix $[\lambda]$, which is already a Jordan canonical form.)

Now suppose that there exist Jordan bases for linear transformations on subspaces of $\mathbb{C}^n$ of dimension less than n, and let $T(\mathbf{z}) = A\mathbf{z}$ for $\mathbf{z} \in \mathbb{C}^n$ and an $n \times n$ matrix A. As noted, we can assume that zero is an eigenvalue of A. Then $\text{rank}(A) < n$; let $r = \text{rank}(A)$. Now T maps $\mathbb{C}^n$ onto the column space of A that is of dimension $r < n$. Let T' be the induced linear transformation of the column space of A into itself, defined by $T'(\mathbf{v}) = T(\mathbf{v})$ for $\mathbf{v}$ in the column space of A. By our induction hypothesis, there is a Jordan basis

$$B' = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r)$$

for this column space of A.

Let S be the intersection of the column space and the nullspace of A. We wish to separate the vectors in B' that are in S from those that are not. The nonzero vectors in S are precisely the eigenvectors in the column space of A with corresponding eigenvalue 0; that is, they are the eigenvectors of T' with eigenvalue 0. In other words, S is the nullspace of T'. Let J' be the matrix representation of T' relative to B'. Because J' is a Jordan canonical form, we see that the nullity of T' (and of J') is precisely the number of zero rows in J'. This is true because J' is an upper-triangular square matrix; it can be brought to echelon form by means of row exchanges that place the zero rows at the bottom while sliding the nonzero rows up. Thus, if $\dim(S) = s$, there are s zero rows in J'. Now in J' we have exactly one zero row for each Jordan block corresponding to the eigenvalue 0—namely, the row containing the bottom row of the block. Because the number of such blocks is equal to the number of strings in B' ending in S, we conclude that there are s such strings. Some of these strings may be of length 1 whereas others may be longer.

Figure 9.11 shows one possible situation when $s = 2$, where two vectors in S—namely, $\mathbf{u}_1$ and $\mathbf{u}_4$—are ending points of strings

$$\mathbf{u}_3 \to \mathbf{u}_2 \to \mathbf{u}_1 \to \mathbf{0} \quad \text{and} \quad \mathbf{u}_5 \to \mathbf{u}_4 \to \mathbf{0}$$

lying in the column space of A. These s strings of B' that end in S start at s vectors in the column space of A; these are the vectors $\mathbf{u}_3$ and $\mathbf{u}_5$ in Figure 9.11.

Because the vector at the beginning of the jth string is in the column space of A, it must have the form $A\mathbf{w}_j$ for some vector $\mathbf{w}_j$ in $\mathbb{C}^n$. Thus we obtain the vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_s$ illustrated in Figure 9.11 for $s = 2$.

Finally, the nullspace of A has dimension $n - r$, and we can expand the set of s independent vectors in S to a basis for this nullspace. This gives rise to $n - r - s$ more vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-r-s}$. Of course, each $\mathbf{v}_i$ is an eigenvector with corresponding eigenvalue 0.

We claim that

$$(\mathbf{u}_1, \ldots, \mathbf{u}_r, \mathbf{w}_1, \ldots, \mathbf{w}_s, \mathbf{v}_1, \ldots, \mathbf{v}_{n-r-s})$$

can be reordered to become a Jordan basis B for A (and of course for T). We reorder it by moving the vectors $\mathbf{w}_i$, tucking each one in so that it starts the appropriate string in B' that was used to define it. For the situation in Figure 9.11, we obtain

$$(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{w}_1, \mathbf{u}_4, \mathbf{u}_5, \mathbf{w}_2, \mathbf{u}_6, \ldots, \mathbf{u}_r, \mathbf{v}_1, \ldots, \mathbf{v}_{n-r-2})$$

as Jordan basis. From our construction, we see that B is a Jordan basis for A if it is a basis for $\mathbb{C}^n$. Because there are $r + s + (n - r - s) = n$ vectors in all, we need only show that they are independent.

Suppose that

$$\sum_{i=1}^{r} a_i\mathbf{u}_i + \sum_{j=1}^{s} c_j\mathbf{w}_j + \sum_{k=1}^{n-r-s} d_k\mathbf{v}_k = \mathbf{0}. \tag{6}$$

Because the vectors $\mathbf{v}_k$ lie in the nullspace of A, if we apply A to both sides of this equation, we obtain

$$\sum_{i=1}^{r} a_iA\mathbf{u}_i + \sum_{j=1}^{s} c_jA\mathbf{w}_j = \mathbf{0}. \tag{7}$$

Because each $A\mathbf{u}_i$ is either of the form $\lambda\mathbf{u}_i$ or of the form $\lambda\mathbf{u}_i + \mathbf{u}_{i-1}$, we see that the first sum is a linear combination of vectors $\mathbf{u}_i$. Moreover, these vectors

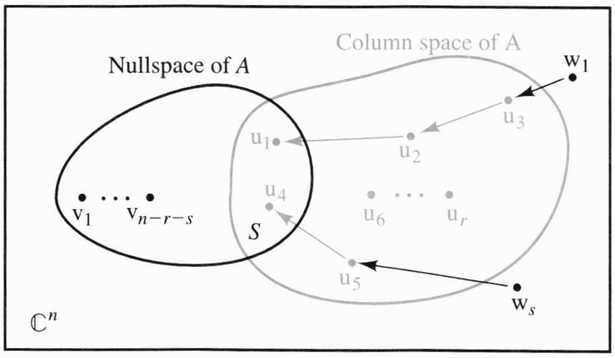

FIGURE 9.11
Construction of a Jordan basis for A ($s = 2$).

$A\mathbf{u}_i$ *do not begin* any string in B'. Now the vectors $A\mathbf{w}_j$ in the second sum are vectors $\mathbf{u}_i$ that appear at the *start* of the s strings in B' that end in S. Thus they do not appear in the first sum. Because B' is an independent set, all the coefficients c_j in Eq. (7) must be zero. Equation (6) can then be written as

$$\sum_{i=1}^{r} a_i\mathbf{u}_i = \sum_{k=1}^{n-r-s} -d_k\mathbf{v}_k. \tag{8}$$

Now the vector on the left-hand side of this equation lies in the column space of A, whereas the vector on the right-hand side is in the nullspace of A. Consequently, this vector lies in S and is a linear combination of the s basis vectors $\mathbf{u}_i$ in S. Because the $\mathbf{v}_k$ were obtained by extending these s vectors to a basis for the nullspace of A, the vector $\mathbf{0}$ is the only linear combination of the $\mathbf{v}_k$ that lies in S. Thus, the vector on both sides of Eq. (8) is $\mathbf{0}$. Because the $\mathbf{v}_k$ are independent, we see that all d_k are zero. Because the $\mathbf{u}_i$ are independent, it follows that the a_i are all zero. Therefore, B is an independent set of n vectors and is thus a basis for $\mathbb{C}^n$. We have seen that, by our construction, it must be a Jordan basis. This completes the induction part of our proof, demonstrating the existence of a Jordan canonical form for every square matrix A.

Our work prior to this theorem makes clear that the Jordan blocks constituting a Jordan canonical form for A are completely determined by the ranks of the matrices $(A - \lambda I)^k$ for all eigenvalues λ of A and for all positive integers k. Thus, a Jordan canonical form J for A is unique except as to the order in which these blocks appear along the diagonal of J. ▲

SUMMARY

1. A Jordan block is a square matrix with all diagonal entries equal, all entries immediately above diagonal entries equal to 1, and all other entries equal to 0.
2. Properties of a Jordan block are given in Theorem 9.8.
3. A square matrix is a Jordan canonical form if it consists of Jordan blocks placed corner to corner along its main diagonal, with entries elsewhere equal to 0.
4. A Jordan basis (see Definition 9.8) for an $n \times n$ matrix A gives rise to a Jordan canonical form J that is similar to A.
5. A Jordan canonical form similar to an $n \times n$ matrix A can be computed if we know the eigenvalues λ_i of A and if we know the rank of $(A - \lambda_i I)^k$ for each λ_i and for all positive integers k.
6. Every square matrix has a Jordan canonical form; that is, it is similar to a Jordan canonical form.

EXERCISES

In Exercises 1–6, determine whether the given matrix is a Jordan canonical form.

1. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

3. $\begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

4. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

5. $\begin{bmatrix} i & 1 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

6. $\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

In Exercises 7–10:

a) Find the eigenvalues of the given matrix J.

b) Give the rank and nullity of $(J - \lambda)^k$ for each eigenvalue λ of J and for every positive integer k.

c) Draw schemata of the strings of vectors in the standard basis arising from the Jordan blocks in J.

d) For each standard basis vector $\mathbf{e}_k$, express $J\mathbf{e}_k$ as a linear combination of vectors in the standard basis.

7. $\begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$

8. $\begin{bmatrix} i & 0 & 0 & 0 & 0 \\ 0 & i & 1 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}$

9. $\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

10. $\begin{bmatrix} i & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

In Exercises 11–14, find a Jordan canonical form for A from the given data.

11. A is 5×5, $A - 3I$ has nullity 2, $(A - 3I)^2$ has nullity 3, $(A - 3I)^3$ has nullity 4, $(A - 3I)^k$ has nullity 5 for $k \geq 4$.

12. A is 7×7, $A + I$ has nullity 3, $(A + I)^k$ has nullity 5 for $k \geq 2$; $A + iI$ has nullity 1, $(A + iI)^j$ has nullity 2 for $j \geq 2$.

13. A is 8×8, $A - I$ has nullity 2, $(A - I)^2$ has nullity 4, $(A - I)^k$ has nullity 5 for $k \geq 3$; $(A + 2I)^j$ has nullity 3 for $j \geq 1$.

14. A is 8×8; $A + iI$ has rank 4, $(A + iI)^2$ has rank 2, $(A + iI)^3$ has rank 1, $(A + iI)^k = O$ for $k \geq 4$.

In Exercises 15–22, find a Jordan canonical form and a Jordan basis for the given matrix.

15. $\begin{bmatrix} -10 & 4 \\ -25 & 10 \end{bmatrix}$

16. $\begin{bmatrix} 5 & -4 \\ 9 & -7 \end{bmatrix}$

17. $\begin{bmatrix} 4 & 0 & 0 \\ 2 & 1 & 3 \\ 5 & 0 & 4 \end{bmatrix}$

18. $\begin{bmatrix} -3 & 0 & 1 \\ 2 & -2 & 1 \\ -1 & 0 & -1 \end{bmatrix}$

19. $\begin{bmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$

20. $\begin{bmatrix} i & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & -1 & 0 & 2 \end{bmatrix}$