Linear algebra 2: exercises for Section 9

Ex. 9.7. Let A be an orthogonal $n \times n$ matrix with entries in $\mathbb{R}$. Show that $\det A = \pm 1$. If A is be an orthogonal 2×2 matrix with entries in $\mathbb{R}$ and $\det A = 1$, show that A is a rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for some $\theta \in \mathbb{R}$.

Ex. 9.8. For which values of $\alpha \in \mathbb{C}$ is the matrix $\begin{pmatrix} \alpha & \frac{1}{2} \\ \frac{1}{2} & \alpha \end{pmatrix}$ unitary?

Ex. 9.9. Let V be the vector space of continuous complex-valued functions defined on the interval $[0,1]$, with the inner product $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$. Show that the set $\{x \mapsto e^{2\pi i k x} : k \in \mathbb{Z}\} \subset V$ is orthonormal. Is it a basis of V?

Ex. 9.10. Show that the matrix of a normal transformation of a 2-dimensional real inner product space with respect to an orthonormal basis has one of the forms

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \alpha & \beta \\ \beta & \delta \end{pmatrix}.$$

Ex. 9.11. Let V be the vector space of infinitely differentiable functions $f\colon \mathbb{R} \to \mathbb{C}$ satisfying $f(x+2) = f(x)$ for all $x \in \mathbb{R}$. Consider the inner product on V given by $\langle p, q\rangle = \int_{-1}^1 p(x)\overline{q(x)}dx$. Show that the operator $D : p \mapsto p''$ is self-adjoint.

Ex. 9.12. Let n be a positive integer. Show that there exists an orthogonal antisymmetric $n \times n$-matrix with real coefficients if and only if n is even.

Ex. 9.13. Consider $\mathbb{R}^n$ with the standard inner product, and let $V \subset \mathbb{R}^n$ be a subspace. Let A be the $n \times n$-matrix of orthogonal projection on V. Show that A is symmetric.