Modular forms of level one, basic examples

Definitions

An element of M_k (resp. S_k) is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ satisfying

$$\begin{aligned} &f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad \text{for all } \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } z \in \mathfrak{H}, \\ &|f(z)| = \mathcal{O}(1) \text{ (resp. } f(z) \to 0) \quad \text{for } \Im z \to \infty. \end{aligned}$$

We have a q-expansion

$$f(z) = \sum_{n=0}^{\infty} a_n(f) q^n \quad \text{where } q = \exp(2\pi i z),$$

here $f \in S_k$ iff $a_0 = 0$.

Eisenstein series

For even $k \geq 4$, we define

$$G_k(z) \;=\; \frac{(k-1)!}{2(2\pi i)^k} \sum_{m,n\in\mathbb{Z}}{}' \frac{1}{(mz+n)^k} \;=\; -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \;\in M_k(\mathrm{SL}_2(\mathbb{Z})).$$

Examples:

$$\begin{aligned} G_4 &= \frac{1}{240} + q + 9q^2 + 28q^3 + 73q^4 + 126q^5 + 252q^6 + 344q^7 + 585q^8 + \cdots \\ G_6 &= -\frac{1}{504} + q + 33q^2 + 244q^3 + 1057q^4 + 3126q^5 + 8052q^6 + 16808q^7 + \cdots \\ G_8 &= \frac{1}{480} + q + 129q^2 + 2188q^3 + 16513q^4 + 78126q^5 + 282252q^6 + \cdots \end{aligned}$$

Delta function

$$\Delta(z) \;=\; q \prod_{n=1}^{\infty} (1-q^n)^{24} \;=:\; \sum_{n=1}^{\infty} \tau(n) q^n \;\in S_{12}(\mathrm{SL}_2(\mathbb{Z})).$$

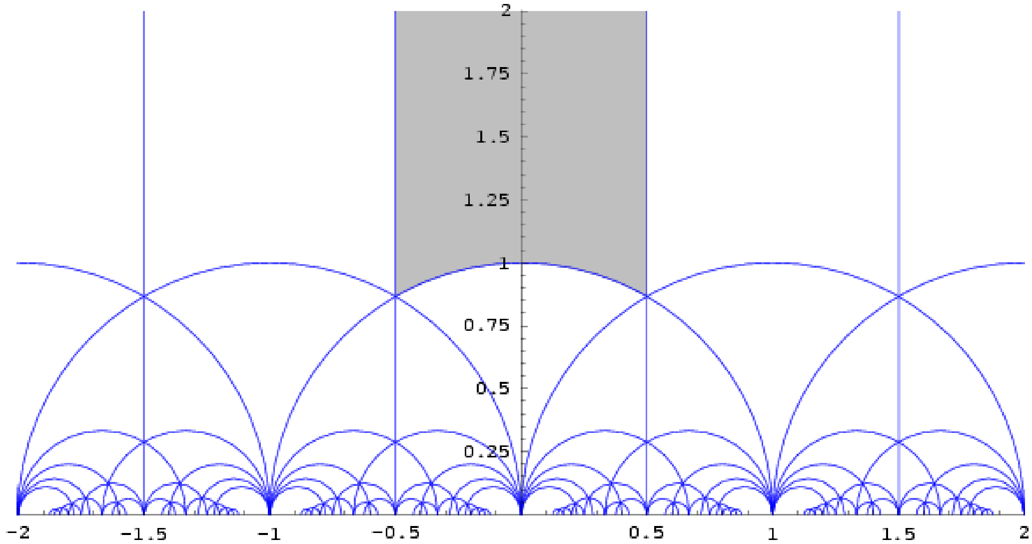
The q-series starts with

$$\Delta \;=\; q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + 84480q^8 + \cdots$$

The function $\tau(n)$ satisfies the properties

$$\begin{array}{rcll} \tau(mn) &=& \tau(m)\tau(n) & \text{if } \gcd(m,n) = 1, \\ \tau(p^{r+1}) &=& \tau(p)\tau(p^r) - p^{11}\tau(p^{r-1}) & \text{for } p \text{ prime and } r \geq 1, \\ |\tau(p)| &\leq& 2p^{11/2} & \text{for } p \text{ prime.} \end{array}$$

Picture of upper half plane



Richard Pink's bookcase: