Brauer groups of fields – Erik Visse
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These are notes from the seminar on obstructions to local-global principles held in Leiden in the spring of 2017. The website for the seminar can be found at http://pub.math.leidenuniv.nl/~vissehd/BMseminar/.

Much of these notes is based on parts of the book by Bjorn Poonen [Poo17].

1 Central simple algebras

Definition 1.1. Let $K$ be a field. A finite-dimensional $K$-algebra $A$ is called

- **central** if the centre of $A$ is $K$, and
- **simple** if $A$ is non-trivial and its only two-sided ideals are 0 and $A$ itself.

Example 1.2. Any finite-dimensional division algebra $D$ is central and simple over its centre $Z(D)$.

Example 1.3. Let $D$ be a finite-dimensional division algebra considered as an algebra over its centre. For any integer $n \geq 1$ the matrix algebra $\text{Mat}_n(D)$ is central and simple over $Z(D)$.

Example 1.3 is fundamental as the following theorem shows:

Theorem 1.4 (Artin–Wedderburn). Let $A$ be a central simple algebra over $K$. Then there exist a division algebra $D$ over $K$ and a positive integer $n$ such that there is an isomorphism $A \cong \text{Mat}_n(D)$. These $D$ and $n$ are uniquely determined up to isomorphism of $D$.

Proof. This is Theorem 2.1.3 from [GS06].

Proposition 1.5. Let $A \neq 0$ be a finite-dimensional $K$-algebra and let $\text{End}_K(A)$ be the $K$-algebra of vector space endomorphisms of $A$. Then $A$ is a central simple algebra if and only if

$\phi : A \otimes_K A^{\text{opp}} \rightarrow \text{End}_K(A),$

$\quad a \otimes b \mapsto (x \mapsto axb).$

is an isomorphism of $K$-algebras.

Proof. This can be deduced from [GS06, Lemma 2.1.6].

Notation 1.6. For a field $K$ we will denote some chosen algebraic closure by $\overline{K}$.

Corollary 1.7. Let $L/K$ be a field extension. Then a finite-dimensional algebra $A$ over $K$ is a central simple algebra over $K$ if and only if $A \otimes_K L$ is a central simple algebra over $L$. Moreover, $A$ is a central simple algebra over $K$ if and only if for some integer $n$ we have $A \otimes_K \overline{K} \cong \text{Mat}_n(\overline{K})$.

Proof. The proof is rather direct using Proposition 1.5. The last assertion follows since the only division algebra over an algebraically closed field $\overline{K}$ is $\overline{K}$ itself.
The last assertion of this corollary implies the interesting fact that the dimensions $[A : K] = [A \otimes_K \overline{K} : \overline{K}]$ are squares.

We will now define the reduced norm map on any central simple algebra, which we will later use to prove that some Brauer groups are trivial.

Let $A$ be any central simple algebra over a perfect field $K$. Then there is an isomorphism $i : A \otimes_K \overline{K} \rightarrow \text{Mat}_n(\overline{K})$. This isomorphism is not unique, but it is up to a $\overline{K}$-algebra automorphisms of $\text{Mat}_n(\overline{K})$, i.e. up to conjugation by an element of $\text{GL}_n(\overline{K})$, so if we compose $i$ with the determinant map $\det : \text{Mat}_n(\overline{K}) \rightarrow \overline{K}$, we get a well-defined map $\text{nr}_{A/\overline{K}} : A \otimes_K \overline{K} \rightarrow \overline{K}$.

The Galois group $\Gamma = \text{Gal}(\overline{K}/K)$ acts on both $A \otimes_K \overline{K}$ and $\text{Mat}_n(\overline{K})$, so for any $\sigma \in \Gamma$ there is an induced map $\sigma i$ and hence an induced map $\sigma \text{nr}_{A/\overline{K}} : A \otimes_K \overline{K} \rightarrow \overline{K}$, which by the argument in the previous paragraph is Galois equivariant. Hence $\text{nr}_{A/\overline{K}}$ restricts to a map $\text{nr}_{A/K} : A \rightarrow K$ which restricts further to a homomorphism $A^\times \rightarrow K^\times$.

**Definition 1.8.** The map $\text{nr}_{A/K} : A \rightarrow K$ constructed above is called the **reduced norm map** of $A$.

## 2 Brauer groups of fields

The last assertion of Corollary 1.7 tells us that any central simple algebra is of the basic form from Theorem 1.4 after passing to an algebraically closed field. This allows us to take equivalence classes. Moreover, passing to an algebraic closure shows that if $A$ and $B$ are two central simple algebras over $K$ then so is $A \otimes_K B$.

**Definition 2.1.** Let $A \cong \text{Mat}_n(D)$ and $A' \cong \text{Mat}_{n'}(D')$ be two central simple algebras over a field $K$ where $D$ and $D'$ are division algebras. Then $A$ and $A'$ are called **similar** if $D \cong D'$ holds.

The **Brauer group** $\text{Br} K$ of $K$ is the set of similarity classes of central simple algebras over $K$ with multiplication $\otimes_K$. The unit class is $[\text{Mat}_n(K)]$ and the inverse of a class $[A]$ is $[A^\text{opp}]$.

### 2.1 Examples

**Example 2.2.** In this section we will treat the following examples.

- Let $K$ be an algebraically closed field. Then $\text{Br}(K) = 0$.
- Finite fields have trivial Brauer group.
- Function fields of curves over algebraically closed fields have trivial Brauer group.
- $\text{Br}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$.
- Let $k$ be a number field and $v$ a non-archimedean place. Then $\text{Br}(k_v)$ is isomorphic to $\mathbb{Q}/\mathbb{Z}$.
Definition 2.3. For $r \in \mathbb{R}_{\geq 0}$, we call a field $K$ a $C_r$-field if every homogeneous polynomial in $n$ variables of degree $d$ with $n > d^r$ has a non-trivial zero.

Remark 2.4. If a field is $C_r$ for some $r$ then it is also $C_R$ for every $R \geq r$.

Example 2.5.
- Being a $C_0$-field is equivalent to being algebraically closed.
- For any $r \geq 0$ the field $\mathbb{R}$ is not $C_r$; the example $x_1^2 + \ldots + x_n^2$ for $n \geq 1$ shows this.

Proposition 2.6. The Brauer group of a $C_1$-field is trivial.

Proof. Let $K$ be a $C_1$-field. For any finite-dimensional central division algebra $D$ over $K$, let $d^2$ be its dimension. We will construct a polynomial that violates the definition of a $C_1$ field unless $d = 1$ holds and hence $D = K$.

A reduced norm form associated to $D$ is a homogeneous polynomial in $d^2$ variables and has degree $d$ (since it comes from the determinant of an $(d \times d)$-matrix). Because the reduced norm restricts to a homomorphism $D^\times \to K^\times$ and $D$ is a division algebra, the norm form has no non-trivial zeroes. This contradicts $K$ being a $C_1$-field unless $d = 1$ holds. Hence the only possibility for $D$ is $K$ itself and thus $Br(K)$ is trivial.

From this proposition, one may quickly prove some of the examples above. We immediately notice that the Brauer group of an algebraically closed field is trivial. In fact, we have already used this by arguing in the proof of Corollary 1.7 that the only division algebra over an algebraically closed field is the field itself.

Proposition 2.7. The Brauer group of a finite field is trivial.

Proof. The Chevalley–Warning theorem says (a bit more than) that any finite field is $C_1$ and hence has trivial Brauer group by Proposition 2.6.

Theorem 2.8 (Tsen’s Theorem). Let $K$ be an algebraically closed field and let $C$ be a curve over $K$, with function field $\kappa$. Then $Br(\kappa)$ is trivial.

Proof. It can be proven that if $k$ is a $C_r$-field then $k(t)$ is a $C_{r+1}$-field and that any algebraic extension of a $C_r$-field is also $C_r$. Hence $\kappa$ is a $C_1$-field and its Brauer group is trivial by Proposition 2.6.

We will leave the Brauer group of $\mathbb{R}$ for Section 3 and the Brauer group of a completion of a number field for Section 4.

3 Other descriptions of the Brauer group

There are several different descriptions of the Brauer group of a field. In this section we will treat two of them.
3.1 The cohomological Brauer group

**Proposition 3.1.** For every perfect field $K$ and every a positive integer $r$ there is a bijection

\[
\{\text{Central simple } K\text{-algebras of dimension } r^2\} \to H^1(\text{Gal}(\overline{K}/K), \text{PGL}_r(\overline{K})).
\]

**Proof.** Let $A$ be any central simple algebra over $K$ of dimension $r^2$. Let $\phi : \text{Mat}_r(\overline{K}) \to A \otimes_K \overline{K}$ be an isomorphism. We have already seen that $\Gamma = \text{Gal}(\overline{K}/K)$ acts on both $A \otimes_K \overline{K}$ and $\text{Mat}_r(\overline{K})$, so every $\sigma \in \Gamma$ induces some map $\sigma \phi$ and hence an element

\[
\xi_\sigma = \phi^{-1} \circ \sigma \phi \in \text{Aut}(\text{Mat}_r(\overline{K})) \cong \text{PGL}_r(\overline{K})
\]

where $\text{Aut}$ stands for the automorphism group as $K$-algebras.

If $\sigma$ and $\tau$ are two elements of $\Gamma$, then an easy calculation shows $\xi_{\sigma \tau} = \xi_\sigma \circ \sigma \xi_\tau$, i.e. $\xi$ defines a 1-cocyle in $\text{PGL}_r(\overline{K})$. One should check that changing $A$ to some $A'$ gives a 1-cocyle $\xi'$ which differs from $\xi$ by a co-boundary element. Hence $\xi$ gives a well-defined element of $H^1(\Gamma, \text{PGL}_r(\overline{K}))$ and the assignment $[A] \mapsto [\xi]$ is injective.

Surjectivity of the map is more difficult and involves (an elementary case of) descent theory, in particular $\text{Mat}_r$ is a quasi-projective group scheme. One proves surjectivity (or even directly bijectivity) for algebraic extensions and then takes the limit on both sides. \hfill $\Box$

Taking the defining exact sequence of $\text{PGL}_r(\overline{K})$, namely

\[
1 \to K^\times \to \text{GL}_r(\overline{K}) \to \text{PGL}_r(\overline{K}) \to 1
\]

and taking the long exact sequence in Galois cohomology, one finds a map of pointed sets

\[
H^1(\Gamma, \text{PGL}_r(\overline{K})) \to H^2(\Gamma, K^\times).
\]

(1)

The following theorem is of importance.

**Theorem 3.2.** The composition of the map from Proposition 3.1 with \((1)\) sending a central simple algebra to its associated element in $H^2(\Gamma, K^\times)$ is an isomorphism.

**Proof.** This is the content of [Poo17, Theorem 1.5.12]. \hfill $\Box$

For a perfect field $K$, the cohomology group $H^2(\Gamma, K^\times)$ is equal to $H^2_{\text{et}}(\text{Spec}(K), G_m)$ and this generalizes to the cohomological Brauer group of a scheme; see next week’s lecture.

**Proposition 3.3.** The Brauer group of $\mathbb{R}$ is of order 2.

**Proof.** Since the absolute Galois group of $\mathbb{R}$ is cyclic, we can use results from class field theory to calculate the cohomology group

\[
H^2(\text{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times) = (\mathbb{C}^\times)^{\text{Gal}(\mathbb{C}/\mathbb{R})} / \text{Nm}(\mathbb{C}^\times) = \mathbb{R}^\times / \mathbb{R}^\times_{>0} \cong \mathbb{Z}/2\mathbb{Z}.
\]

$\Box$
3.2 Severi–Brauer varieties

The cohomological description above works because in general the cohomology group $H^1(\Gamma, \text{Aut}(X_K))$ parametrizes $\overline{K}$-twists of $X$. This statement lies at the heart of the descent theory that we mentioned in the previous subsection. The group $\text{PGL}_n(\overline{K})$ is the automorphism group of an object other than the matrix group: namely $\mathbb{P}^{r-1}_K$.

**Definition 3.4.** A **Severi–Brauer variety** $X$ of dimension $d$ over a perfect field $K$ is a variety such that $X \times \text{Spec}(\overline{K})$ is isomorphic to $\mathbb{P}^d_K$.

Descent theory shows that the group $H^1(\text{Gal}(\overline{K}/K), \text{PGL}_r(\overline{K}))$ parametrizes Severi–Brauer varieties of dimension $r-1$ over $K$. Since the union of these groups ranging over all $r$ is isomorphic to the Brauer group $\text{Br}(K)$, we see that the Brauer group parametrizes Severi–Brauer varieties over $K$ of any dimension.

**Proposition 3.5 (Châtelet).** For $X$ be an $n-1$-dimensional Severi–Brauer variety over a perfect field $K$, the following statements are equivalent:

1. $X \cong \mathbb{P}^{n-1}_K$,
2. $X$ is birational to $\mathbb{P}^{n-1}_K$,
3. $X(K) \neq \emptyset$.

**Proof.** The implication 1 $\Rightarrow$ 2 is trivial; the implication 2 $\Rightarrow$ 3 follows from the Lang–Nishimura theorem. The implication 3 $\Rightarrow$ 1 needs a bit more work.

Assume that there is a point $Q \in X(k)$. Then the pair $(X, Q)$ is a twist of the pair $(\mathbb{P}^{n-1}_K, P = (1 : 0 : \ldots : 0))$. Where the coordinates of $P$ are chosen without loss of generality.

Now the automorphisms of $(\mathbb{P}^{n-1}_K, P)$ over $\overline{K}$ are those of $\mathbb{P}^{n-1}_K$ that fix $P$. They form a subgroup of $\text{PGL}_n(\overline{K})$, namely

$$
\begin{pmatrix}
1 & * & \cdots & *\\
0 & * & \cdots & *\\
\vdots & \vdots & \ddots & \vdots \\
0 & * & \cdots & *
\end{pmatrix}
$$

where we have exploited scaling by elements of $\overline{K}$ to set the first entry to 1. Forgetting the first row and column gives a homomorphism to $\text{GL}_{n-1}(\overline{K})$, so we get an exact sequence

$$
1 \rightarrow \mathbb{K}^{n-1} \rightarrow \text{Aut}(\mathbb{P}^{n-1}_K, P) \rightarrow \text{GL}_{n-1}(\overline{K}) \rightarrow 1
$$

of $\text{Gal}(\overline{K}/K)$-modules. Indeed, this description of the kernel represents the $n-1$ coordinates in the first row. We take the exact sequence in cohomology and realize that the $H^1$ of the first and third term are trivial. Hence the same holds for the middle term. Hence the pair $(\mathbb{P}^{n-1}_K, P)$ has no non-trivial twists, i.e. $X$ was already isomorphic to $\mathbb{P}^{n-1}_K$. \[\square\]

\[1\] If $X \rightarrow Y$ is a rational map between varieties over $k$ and assume that $Y$ is proper. If $X$ has a smooth $k$-point, then $Y(k) \neq \emptyset$.  

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Theorem 3.6 (Châtelet). Severi–Brauer varieties over a number field satisfy the Hasse principle.

Proof. For any Severi–Brauer variety $X$ of dimension $n - 1$ over a number field $k$, let $[X]$ denote its representing class in $\text{Br}(k)$.

By Proposition 3.5 we have $X(k) \neq \emptyset \iff X \cong \mathbb{P}^{n-1}_k$ and the latter is equivalent to $[X] = 0$.

The base-change $X_{k_v}$ is a Severi–Brauer variety over $k_v$; its class $[X_{k_v}]$ in $\text{Br}(k_v)$ is the image of $[X]$ under the map $\text{Br}(k) \rightarrow \text{Br}(k_v)$.

Similar to the global case, we also have $X_{k_v}(k_v) \neq \emptyset \iff [X_{k_v}] = 0$. Theorem 4.2 in the next section will show that the map $\text{Br}(k) \rightarrow \bigoplus_v \text{Br}(k_v)$ is injective, which proves the theorem. \qed

4 The Hasse invariant and Hilbert norm residue symbol

4.1 The Hasse invariant map

Theorem 4.1. Let $k$ be a number field. For every place $v$ of $k$, there exists an injective homomorphism $\text{inv}_v : \text{Br}(k_v) \rightarrow \mathbb{Q}/\mathbb{Z}$ called the Hasse invariant whose image is

$$\text{inv}_v(\text{Br}(k_v)) = \begin{cases} \frac{1}{2}\mathbb{Z}/\mathbb{Z} & \text{if } k_v = \mathbb{R}, \\ 0 & \text{if } k_v = \mathbb{C}, \\ \mathbb{Q}/\mathbb{Z} & \text{if } v \text{ is non-archimedean}. \end{cases}$$

Sketch of proof. The statements for $\text{Br}(\mathbb{R})$ and $\text{Br}(\mathbb{C})$ follow from the determination of these groups. The statement for $\text{Br}(k_v)$ with $v$ non-archimedean is more complicated and involves a large portion of local class field theory. We will not repeat the details here, but it involves the following: for each cyclic extension $L \supset k_v$, one may construct so-called cyclic algebras which turn out to be central and simple. A cyclic algebra can be given by just the data of a continuous character $\chi : \text{Gal}(k_v/k_v) \rightarrow \mathbb{Q}/\mathbb{Z}$ and an element $b \in k_v^\times$.

Taking a step back, we prove that for every unramified Galois extension $F \supset k_v$ every central simple algebra over $k_v$ that after field extension is isomorphic to $\text{Mat}_n(F)$ is in fact a cyclic algebra.

Then, proving that every central simple algebra becomes a matrix algebra after base-change to an unramified extension $F$, one can define $\text{inv}_v$ for any class in $\text{Br}(k_v)$ by just defining it for cyclic algebras.

The Hasse invariant of a cyclic algebra $(\chi, b)$ is $\text{inv}_v(\chi, b) = \nu(b)\chi(\text{Frob}_F/k_v)$.

Of course, one is then left to prove that this is independent of all choices made. \qed

2Remember that for an unramified extension of local fields, the Galois group is isomorphic to the Galois group of the residue field extension, which is cyclic and generated by Frobenius.
For any field extension $K \subset L$ there is a natural map $\text{Br}(K) \to \text{Br}(L)$ by taking the tensor product $\otimes_K L$. This is of particular importance for completions of number fields $k \subset k_v$.

**Theorem 4.2.** For any number field $k$ there is an exact sequence

$$0 \to \text{Br}(k) \to \bigoplus_{v \in \Omega_k} \text{Br}(k_v) \xrightarrow{\sum_{v \in \Omega_k} \text{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0.$$ 

**Proof.** The is part of [Poo17, Theorem 1.5.36] which refers to [Tat67].

This theorem will lie at the heart of the Brauer–Manin obstruction that we will define in a later lecture.

### 5 Fun stuff

#### 5.1 Quaternion algebras

**Definition 5.1.** Let $K$ be a field of characteristic different from 2 containing two non-zero elements $a, b$. The quaternion algebra $K_{a,b}$ is the four-dimensional vector space with basis $\{1, i, j, k\}$ satisfying the multiplication rules

$$i^2 = a, \quad j^2 = b, \quad ij = k = -ji.$$ 

A quaternion algebra over $K$ is an algebra that is isomorphic to $K_{a,b}$ for some $a, b \in K^\times$.

**Proposition 5.2.** Any quaternion algebra is central and simple. It is of order 2 in the Brauer group.

**Proof.** That a quaternion algebra is central and simple is easily checked. The second statement is [GS06, Cor. 1.5.3].

Although we only defined quaternion algebras over fields of characteristic different from 2, people often call any 4-dimensional central simple algebra over a field a quaternion algebra. Away from characteristic 2, those have a structure as described.

**Example 5.3.** The usual Hamilton quaternion algebra is $\mathbb{H} = \mathbb{R}_{-1,-1}$. The corresponding Severi–Brauer variety is the conic $x^2 + y^2 + z^2 = 0$.

**Example 5.4.** More generally, the Severi–Brauer variety that corresponds to $\mathbb{Q}_{v,a,b}$ is given by $ax^2 + by^2 = z^2$.

#### 5.2 The Hilbert norm residue symbol

Let $K = k_v$ be a non-archimedean completion of a number field and let $n$ be a positive integer such that $K$ contains the full group of units $\mu_n(K)$. Choose some
generator $\zeta$ for $\mu_n(K)$. Then for any $a \in K^\times$ and $x^n = a$ we define a character

$$\chi_a : \text{Gal}(\overline{K}/K) \longrightarrow \text{Gal}(K(x)/K) \longrightarrow \frac{1}{n}\mathbb{Z}/\mathbb{Z},$$

$$(x \mapsto \zeta x) \mapsto \frac{1}{n} \mod \mathbb{Z}.$$  

For any $b \in K^\times$, one can define the cyclic algebra $(\chi_a, b)$ which gives an element of $\text{Br}(K)$.

This data is enough to define the $n$th Hilbert norm residue symbol.

**Definition 5.5.** The $n$th Hilbert norm residue symbol of $K$ is the function

$$\{ , \}_n : K^\times \times K^\times \rightarrow \mu_n(K),$$

$$(a, b) \mapsto \zeta^{n \text{inv}_v(\chi_a, b)}.$$  

**Remark 5.6.** The name “norm residue symbol” comes from the fact that the symbol is invariant under multiplying $b$ by a norm from $K(\sqrt[n]{a})$. In fact, it satisfies many more interesting properties.

There is a nice link with solving polynomial equations. For $n = 2$ and $k = \mathbb{Q}$, the Hilbert norm residue symbol as defined here has an other interpretation.

**Theorem 5.7.** For any $v \in \Omega_{\mathbb{Q}}$ and $a, b \in \mathbb{Q}^\times_v$ consider the curve $C$ given by

$$ax^2 + by^2 = z^2.$$  

Then $C(\mathbb{Q}_v)$ is non-empty if and only if $\{a, b\}_{\mathbb{Q}_v, 2} = 1$ holds.

**Proof.** This is implied by Proposition 3.5 and the fact that the quaternion algebra $K_{a,b}$ is isomorphic to the cyclic algebra $(\chi_a, b)$ constructed above.

**Notation 5.8.** If $K = \mathbb{Q}_v$ and $n = 2$ we write $(a, b)_v = \{a, b\}_{\mathbb{Q}_v, 2}$. In the literature one often also encounters this notation for $\text{inv}_v(\mathbb{Q}_v, a, b)$, so one should be careful in checking which symbol is meant. Of course, the one can be translated into the other.

### 5.3 Quadratic reciprocity

Taking $n = 2$ and letting $K$ range over all $\mathbb{Q}_v$ (including $\mathbb{R}$), we obtain a more general version of a well-known theorem.

**Theorem 5.9 (Quadratic reciprocity).** For any two $a, b \in \mathbb{Q}^\times_v$, the following equality holds.

$$\prod_{v \in \Omega_{\mathbb{Q}}} (a, b)_v = 1.$$  

**Proof.** From $\{a, b\}_{\mathbb{Q}_v, 2} = (a \circ \text{inv}_v)(k_{v,a,b})$ and Theorem 4.2 we immediately conclude that the statement holds.  

\[\text{cf. the proof of Theorem } 4.1\]
Stated in this form, and comparing with Theorem 5.7, this is a base step in proving that quadratic hypersurfaces satisfy the Hasse principle, but we won’t treat the details in this seminar.

Let’s finish this talk by explaining why we call this a generalization of quadratic reciprocity.

The values of the symbols below (in particular the proof that we need only consider the four values \( v \in \{2, p, q, \infty\} \) can be found in [Cas78, p. 43 & 44].

Let \( p \) and \( q \) be two different odd primes. We use short-hand notation for the relevant Hilbert norm residue symbols. We interpret the symbol \( \{p, q\}_v \) according to Theorem 5.7 to see that for \( v \notin \{2, p, q, \infty\} \) we have \( \{p, q\}_v = 1 \). We only need to compute the symbol for these four special places.

- We have \( \{p, q\}_2 = (-1)^{(p-1)(q-1)/4} \) since the quadratic equation \( x^2 + y^2 = z^2 \) has a non-trivial solution modulo 2, but it doesn’t modulo 4 if both \( p \) and \( q \) are 3 mod 4.
- We have \( \{p, q\}_p = \left( \frac{2}{q} \right) \) and \( \{p, q\}_q = \left( \frac{2}{p} \right) \).
- We have \( \{p, q\}_\infty = 1 \).

Hence we conclude from Theorem 5.9 that we have

\[
\left( \frac{p}{q} \right) \left( \frac{q}{p} \right) = (-1)^{(p-1)(q-1)/4}.
\]

REFERENCES


