# Kontsevich' formula for rational plane curves – Erik Visse

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These are the notes from the seminar on Moduli stacks of curves held in Leiden in the autumn of 2015. The website for this seminar can be found at `http://pub.math.leidenuniv.nl/~bommelrvan/2015/modstack.htm`.

These notes take heavily (!) after Kock and Vainsencher's explanatory text about Kontsevich' formula for rational plane curves [KV03].

The author thanks David Holmes for suggesting this topic and for his help with the material.

For the whole talk, $d$ will be a positive integer. For notation not introduced in these notes, please refer to the notes from last week's talk, available on the author's website (and that of the seminar).

## 1 Evaluation and forgetful maps

Definition 1.1. For any $1 \leq i \leq n$ the $i$'th *evaluation map* is

$$\begin{array}{rcl} \nu_i \colon \overline{M}_{0,n}(\mathbb{P}^r, d) & \to & \mathbb{P}^r \\ (C; p_1, \ldots, p_n; \mu) & \mapsto & \mu(p_i) \end{array}$$

Lemma 1.2. *These evaluation maps are flat.*

*Proof.* Since each evaluation map is invariant under the action of $\mathrm{Aut}(\mathbb{P}^r)$ and its action on $\mathbb{P}^r$ is transitive, the result follows by Grothendieck's generic flatness result (one needs some familiarity with the actual construction of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ to fill in the details of this proof). □

It will be these evaluation maps that allow us to relate geometry of $\mathbb{P}^r$ to the geometry of $\overline{M}_{0,n}(\mathbb{P}^r, d)$.

Example 1.3. If $H \subset \mathbb{P}^r$ is a hyperplane, then for each $i$ the inverse image $\nu_i^{-1}(H)$ is a divisor in $\overline{M}_{0,n}(\mathbb{P}^r, d)$ consisting of all maps whose $i$'th marked point is mapped into $H$. If $Q \in \mathbb{P}^2$ is a point, then for each $i$ the inverse image $\nu_i^{-1}(Q)$ is of codimension 2 in $\overline{M}_{0,n}(\mathbb{P}^2, d)$.

Another class of important maps is that of forgetful maps.

Definition 1.4. A *forgetful map* of stable curves $\varepsilon \colon \overline{M}_{0,n+1} \to \overline{M}_{0,n}$ is one that omits one mark. If after omitting said mark the curve would not be stable, one needs to 'stabilize', i.e. twigs that become unstable need to be contracted.

Remark 1.5. One may define forgetful maps for $\overline{M}_{0,n+1}(\mathbb{P}^r, d) \to \overline{M}_{0,n}(\mathbb{P}^r, d)$ similarly. The use of the contraction procedure in the above definition yields a well-defined map since stability of maps places no requirement on twigs that aren't mapped to a point.

If $A \subsetneq B$ are two sets of marks, then by composing sufficiently many forgetful maps, one may arrive at a forgetful map $\overline{M}_{0,A} \to \overline{M}_{0,B}$. The order in which the marks are forgotten is unimportant in this procedure.

There is the following fact:

LEMMA 1.6. *The fibres of such forgetful maps are reduced.*

*Proof.* See [KV03] Section 1.4. □

Another class of forgetful maps is those that forget the data of the map to $\mathbb{P}^r$ and only takes the domain into account. By possibly stabilizing image curves, one finds a map for $n \geq 3$

$$\eta_n \colon \overline{M}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,n}.$$

LEMMA 1.7. *For $n \geq 3$, the map $\eta_n$ is flat.*

*Proof.* One may prove this from the construction of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ given in [KV03] but skipped in these talks. The special case $n = 4$ has an alternative and rather easy proof based on the fact $\overline{M}_{0,4} \cong \mathbb{P}^1$ and $\eta_4$ is surjective. See [Har77] Prop.III.9.7. □

REMARK 1.8. Composing the two kinds of forgetful maps, one gets maps

$$\overline{M}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,4} \tag{1}$$

which, are surjective and hence, as in the last proof, flat.

## 2 Boundary divisors

We'll study some divisors of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ that arise from pull-backs of boundary divisors of $\overline{M}_{0,4}$, hence the terminology.

DEFINITION 2.1. For a non-negative integer $d$, a $d$-weighted partition of the set $\{1, \ldots, n\}$ consists of a partition $A \cup B = \{1, \ldots, n\}$ together with a partition $d_A + d_B = d$ into non-negative integers.

Recall that the boundary of $M_{0,n} \subset \overline{M}_{0,n}$ is parametrized by graphs of a certain kind. We find a similar behaviour in here, without further explanation or justification.

FACT 2.2. For any $d$-weighted partition of the marked points

$$A \cup B = \{p_1, \ldots, p_n\} \text{ where } \#A \geq 2 \text{ if } d_A = 0 \text{ and } \#B \geq 2 \text{ if } d_B = 0$$

there exists an irreducible divisor $D(A, B; d_A, d_B)$ of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ called a *boundary divisor*. A general point on this divisor represents a map $\mu$ whose domain is a tree with two twigs $C_A, C_B$ such that all marked points of $A$ lie in $C_A$ and further such that the restriction of $\mu$ to $C_A$ is a map of degree $d_A$, and analogously for $B$.

LEMMA 2.3. *If $n \geq 1$, then the number of boundary divisors of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is $2^{n-1}(d+1) - n - 1$. The number of boundary divisors of $\overline{M}_{0,0}(\mathbb{P}^r, 0)$ is $\lfloor d/2 \rfloor$.*

*Proof.* For positive $n$ this is an easy exercise in combinatorics, treating the cases $d_A = 0$ and $d_A = d$ separately. The case $n = 0$ is just the number of ways one may write $d$ as the sum of two positive integers. □

We'll describe some boundary divisors of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ that are linearly equivalent. In order to do so, we first study boundary divisors of $\overline{M}_{0,4}$. Remark that a rational stable curve with 4 marked points either consists of a single smooth twig (with all marked points on it) or two twigs intersecting once, both with two marked points. The former lie in $M_{0,4}$ so the boundary contains only the latter ones.
The special divisors that we will need are the pull-backs of boundary divisors of $\overline{M}_{0,4}$ along the map (1). Let us denote a boundary divisor $(ij|kl)$ of $\overline{M}_{0,4}$ parametrized by trees consisting of two twigs, say $C_A$ and $C_B$ with marks $i$ and $j$ on $C_A$ and marks $k$ and $l$ on $C_B$, and denote its pull-back by $D(ij|kl)$.

Lemma 2.4. *We have the following equality of divisors*

$$D(ij|kl) = \sum D(A, B; d_A, d_B)$$

*where the sum is taken over all $d$-weighted partitions with $i, j \in A$ and $k, l \in B$.*

*Proof.* Clearly, each such $D(A, B; d_A, d_B)$ in the sum maps to $(ij|kl)$. Since the fibres are reduced (cf. 1.6), the coefficients in this sum are all 1. □

There are only three ways to group marks $\{i, j, k, l\}$ into two groups of two elements, and each of these three ways give linear equivalent divisors (on $\overline{M}_{0,4} \cong \mathbb{P}^1$) since any two points in $\mathbb{P}^1$ are linearly equivalent. Therefore, we arrive at the following very important relation:

Theorem 2.5. *The following three boundary divisors of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ are linearly equivalent, where in each case $A \cup B = [n]$ and $d_A + d_B = d$ hold:*

$$\sum_{\substack{i,j \in A \\ k,l \in B}} D(A, B; d_A, d_B) \equiv \sum_{\substack{i,k \in A \\ j,l \in B}} D(A, B; d_A, d_B) \equiv \sum_{\substack{i,l \in A \\ j,k \in B}} D(A, B; d_A, d_B).$$

## 3 Rational plane curves

We start of with the following well-known fact (Euclid's first postulate).

Fact 3.1. Through any two different points in the plane, there exists a unique line passing through the both of them.

In this section, we will occupy ourselves by counting the number $N_d$ of degree $d$ rational curves through any $3d - 1$ points in the plane *in general position.*[1] Here, general position means any position such that $N_d$ is neither zero not infinity. The result will be a recursive formula for $d$ which uses $d = 1$ as a base case.

[1] For $3d - 3$ or $3d - 2$ points, *general position* will mean any position obtained from $3d - 1$ points in general position having then removed any (two) point(s).

It is still well known (though less so than the previous fact) that $N_2 = 1$ also holds; that is, any general five points in the plane describe a unique conic. We will prove this fact using the machinery developed in these talks in order to understand the proof for general $d$.

Before starting off, we must remark that we will, in fact, count stable maps satisfying some properties. A priori this could mean that we are counting some rational curves multiple times (with different markings). It can be shown that the count that we make actually (also) gives us the number we are looking for. See [KV03] Section 3.5 for a discussion about this.

THEOREM 3.2. *Given any five points in $\mathbb{P}^2$ in general position, there exists a unique conic passing through each of them, i.e. $N_2 = 1$.*

*Proof.* We'll do a computation involving intersection theory in $\overline{M}_{0,6}(\mathbb{P}^2, 2)$ and we will generally indicate six marks by the symbols $m_1, m_2, p_1, \ldots, p_4$ since two of the marks will have a different role than the four others.
Let $L_1$ and $L_2$ be two lines in $\mathbb{P}^2$ and $Q_1, \ldots, Q_4$ be four points in $\mathbb{P}^2$ in general position. Define $Y \subset \overline{M}_{0,6}(\mathbb{P}^2, 2)$ to be the subset of maps

$$(C; m_1, m_2, p_1, \ldots, p_4; \mu) \text{ such that } \begin{cases} \mu(m_1) \in L_1, \\ \mu(m_2) \in L_2, \\ \mu(p_i) = Q_i, \text{ for } i = 1, \ldots, 4. \end{cases}$$

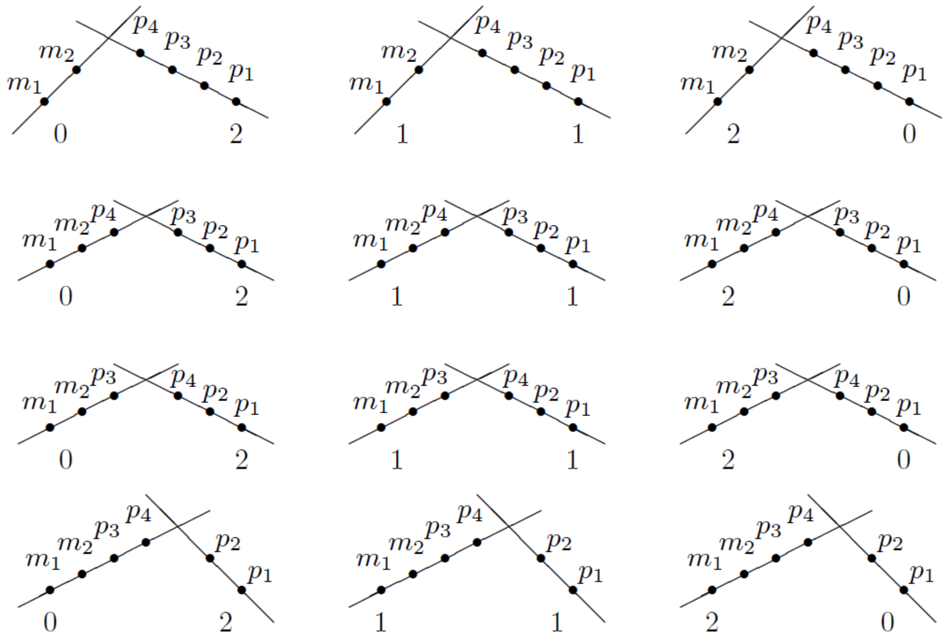
Then $Y$ equals the following intersection

$$Y = \nu_{m_1}^{-1}(L_1) \cap \nu_{m_2}^{-1}(L_2) \cap \nu_{p_1}^{-1}(Q_1) \cap \ldots \cap \nu_{p_4}^{-1}(Q_4).$$

Since all the evaluation maps $\nu$ are flat, the total codimension of $Y \subset \overline{M}_{0,6}(\mathbb{P}^2, 2)$ is 10. Since furthermore the dimension of $\overline{M}_{0,6}(\mathbb{P}^2, 2)$ is 11, $Y$ is a curve and we can compute the intersection of $Y$ with boundary divisors. It is a fact that we will not explain here (which is explained in [KV03] Section 3.4) that $Y$ intersects boundary divisors transversally and that these intersections take place in the smooth locus $\overline{M}^*_{0,6}(\mathbb{P}^2, 2)$.
We consider the map $\overline{M}_{0,6}(\mathbb{P}^2, 2) \to \overline{M}_{\{m_1,m_2,p_1,p_2\}}$ forgetting both the map $\mu$ and the marks $p_3$ and $p_4$. Theorem 2.5 gives

$$Y \cap D(m_1, m_2|p_1, p_2) \equiv Y \cap D(m_1, p_1|m_2, p_2). \tag{2}$$

We will consider the intersections with each of the divisors in the sum on the left-hand side $D(m_1, m_2|p_1, p_2) = \sum D(A, B; d_A, d_B)$ first. There are 12 configurations of degrees $d_A$ and $d_B$ and distribution of the marks $p_3$ and $p_4$ on the two twigs $C_A$ and $C_B$, pictured below.




The twig $C_A$ is always drawn on the left and the numbers represent the degrees $d_A$ and $d_B$.

We'll compute the intersection with $Y$ for each of the diagrams. In each case we will either find some contribution to the sum or no contribution by assumption of the points and lines being general.

In the first column, we have $d_A = 0$, i.e. the twig $C_A$ is mapped to a point, say $z \in \mathbb{P}^2$. Since $C_A$ contains both $m_1$ and $m_2$ which are mapped into the lines $L_1$ and $L_2$ respectively, $z$ is the unique point of intersection of $L_1$ and $L_2$. In the bottom three diagrams there is at least one other mark $p_i$ ($i \in \{3, 4\}$) mapped to $z$, contradicting generality. So $Y$ has empty intersection with each of these divisors. Now for the top diagram in this column. The twig $C_B$ is mapped to a conic containing both each of the four $Q_i$ and the image of the intersection point $C_A \cap C_B$, which is $z$. How many of such conics are there? Well, that is exactly the number $N_2$ that we are looking for!

In the middle column we have $d_A = d_B = 1$ and hence either twig is mapped to a line, which must be two distinct ones by generality. In the first three diagrams we find three of the points $Q_i$ to be collinear – again, excluded by generality. In the fourth diagram, the two lines $\mu(C_A)$ and $\mu(C_B)$ are uniquely determined by them passing through $p_3, p_4$ and $p_1, p_2$ respectively (and by $N_1 = 1$, here is where the recursion appears in the case for general $d$). The number of configurations as in this diagram is 1 since the images of each of the special points (the marks and the node) is determined and each twig has at least three special points.

In the third column the twig $C_B$ is mapped to a point, but as in each of these diagrams the twig contains at least two of the marks $p_i$, at least two of the points $Q_i$ would be equal, contradicting generality. Therefore the diagrams in this column do not contribute to the intersection.

Combining our results so far, we have $Y \cap D(m_1, m_2|p_1, p_2) = N_2 + 1$.

For the right-hand side of (2) we could also draw such diagrams. One would quickly conclude that each diagram where one twig has degree 0 yields a contradiction since that twig contains both $p_i$ and $m_i$ ($i \in \{1, 2\}$) so $Q_i$ would have to lie on $L_i$. In the "middle column" $d_A = d_B = 1$ one cannot have both $p_3$ and $p_4$ on the same twig as then $Q_3$, $Q_4$ and either $Q_1$ or $Q_2$ would be collinear. Therefore the only configurations that remain are $p_3 \in C_A$ and $p_4 \in C_B$ or its mirror image. The number of either of such configuration is the number of ways of drawing a line (degree $d_A = d_B = 1$-curve) through two distinct given points, i.e. 1. Hence this diagram contributes $1 + 1 = 2$ to the sum in the right-hand side of (2).

To sum up, we get $N_2 + 1 = 2$, and hence the desired $N_2 = 1$. □

Theorem 3.3 (Kontsevich). *The following recursive relation holds:*

$$N_d + \sum_{\substack{d_A+d_B=d \\ d_A \geq 1, d_B \geq 1}} \binom{3d-4}{3d_A-1} d_A^3 \cdot d_B \cdot N_{d_A} \cdot N_{d_B} = \sum_{\substack{d_A+d_B=d \\ d_A \geq 1, d_B \geq 1}} \binom{3d-4}{3d_A-2} d_A^2 \cdot d_B^2 \cdot N_{d_A} \cdot N_{d_B}.$$

*Proof.* We take a similar strategy as in the case $d = 2$ and calculate intersections in $\overline{M}_{0,3d}(\mathbb{P}^2, 2)$. Name the $3d =: n$ marks $m_1, m_2, p_1, \ldots, p_{n-2}$. Let $L_1$ and $L_2$ be two lines and $Q_1, \ldots, Q_{n-2}$ to be points, all generally positioned. Let $Y$ be the intersection

$$Y = \nu_{m_1}^{-1}(L_1) \cap \nu_{m_2}^{-1}(L_2) \cap \nu_{p_1}^{-1}(Q_1) \cap \ldots \cap \nu_{p_{n-2}}^{-1}(Q_{n-2}).$$

As before, the intersections of $Y$ and the boundary divisors are transversal and happen in $\overline{M}_{0,n}^*(\mathbb{P}^2, d)$. The result will again follow from the equality

$$Y \cap D(m_1, m_2|p_1, p_2) \equiv Y \cap D(m_1, p_1|m_2, p_2). \tag{3}$$

We examine the left-hand side first. If $d_B = 0$ holds (the "third column"), then concluding that we get no contribution is completely analogous to the case $d = 2$. For $d_A = 0$ we only find a contribution in case where all the $3d-4$ remaining marks lie on $C_B$. And the number of such configurations is $N_d$ – the $3d - 4$ spare marks, combined with $p_1$, $p_2$ and the point of intersection $C_A \cap C_B$ form a set of $3d - 1$ points generally positioned in the plane.

In order for any configuration with $d_A \geq 1$ and $d_B \geq 1$ to contribute, among the $3d-4$ spare marks, we may have at most $3d_B-3$ of them on $C_B$ (since then together with $p_1$ and $p_2$ we have $3d_B - 1$ of them). That means that we need at least $3d_A - 1$ of the spare marks on $C_A$. However, we may also have at most $3d_A - 1$ marks on $C_A$ and we conclude that we only get a contribution when there are *exactly* $3d_A - 1$ of the $3d - 4$ spare marks on $A$ and exactly $3d_B - 1$ on $C_B$. These can be chosen in $\binom{3d-4}{3d_A-1}$ ways. Now for the image of $C_A$ there are $N_{d_A}$ possibilities and $N_{d_B}$ for the image of $C_B$. We retain some freedom in placing the marks $m_1$ and $m_2$ on $C_A$. The mark $m_1$ has to land on the intersection $\mu(C_A) \cap L_1$ which by Bézout's theorem has $d_A$ points and similarly for $m_2$. Finally, again by Bézout's theorem, there are $d_A d_B$

possibilities in placing the unique point in $\mu(C_A) \cap \mu(C_B)$. Hence we arrive at the expression on the left-hand side of (3).

Studing the right-hand side of (3) is again similar to the $d = 2$ case. If either $d_A$ or $d_B$ is zero, that would imply $Q_1 \in L_1$ or $Q_2 \in L_2$, contradicting generality. As above, the only contribution in other cases is when there are exactly $3d_A - 2$ further marks on $C_A$ and $3d_B - 2$ on $C_B$, which can be choosen in $\binom{3d-4}{3d_A-2}$ ways. The images of $C_A$ and $C_B$ can be choosen in $N_{d_A}$ and $N_{d_B}$ ways respectively. There are $d_A$ choices for $\mu(m_1)$ and $d_B$ for $\mu(m_2)$. Again, there are $d_A d_B$ choices for $\mu(C_A) \cap \mu(C_B)$. $\square$

## Bibliography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[KV03] Joachim Kock and Israel Vainsencher. Kontsevich's formula for rational plane curves. available on `http://www.dmat.ufpe.br/~israel/kontsevich.html`, 2003.