COUNTING RATIONAL POINTS ON CERTAIN VARIETIES
Erik Visse

These are the notes for my talk at the Leiden PhD seminar held at October 14th, 2014. The talk was aimed at an audience with a widely diverse mathematical background, ranging from algebraic geometers to statisticians who may not have taken any algebra courses unless one counts some basic linear algebra.

1 Projective space

For geometric purposes, it is often customary to work not in Euclidean space $k^n$ (for $k$ a field) but in what we call ‘projective space’.

**Definition 1.1.** The *projective $n$-space* over a field $k$ is the set of 1-dimensional linear subspaces of the vector space $k^{n+1}$. We denote it by $\mathbb{P}^n(k)$, often also sloppily by $\mathbb{P}^n$ if no confusion about the field $k$ is likely to arise.

We’ll treat some low-dimensional cases to see how projective space gives us an extension of Euclidean space. For purposes of intuition it is OK to think of $k = \mathbb{R}$. It is really helpful to draw pictures for this. (I hope to add good ones to the notes.)

**Example 1.2.** The projective line $\mathbb{P}^1(k)$ is the set of lines through the origin in $k^2$. Let $L$ be a line in $k^2$ that does not go through the origin. Then almost every line through 0 (i.e. elements, or ‘points’ in $\mathbb{P}^1$) intersects the line $L$ in a unique point. Only the element of $\mathbb{P}^1$ that is parallel to $L$ does not intersect it. We can think of the elements of $\mathbb{P}^1$ as the points on the line $L$ together with an added point at infinity. Remark that, even though on $L$ we can travel in two opposite directions, we only get one point at infinity.

**Example 1.3.** The projective plane $\mathbb{P}^2(k)$ can similarly be described by use of a plane $S \subset k^3$ not through the origin where we identify a line through the origin with its intersection point with $S$ if it exists. The lines that do not intersect $S$ lie parallel to it and span a plane. In fact, the set of these lines form a $\mathbb{P}^1$. The projective plane can therefore be thought of as a Euclidean plane together with a line (and another point) at infinity.

**Remark 1.4.** In $\mathbb{P}^2$ we see something interesting happening: every two lines in $\mathbb{P}^2$ intersect; *there is no notion of parallelism*. To see this, one has to consider what a line in $\mathbb{P}^2$ actually is. Each of its points is a line in $k^3$ and the points on a line in $\mathbb{P}^2$ span a plane in $k^3$. And we know from basic linear algebra that any two planes (through the origin) in a three-dimensional space intersect in a line, i.e. a point in $\mathbb{P}^2$. Intuitively, if we have two ‘parallel’ lines in a plane, they intersect at infinity. *And infinity is part of our world.*

**Remark 1.5.** We can describe $\mathbb{P}^n(k)$ also as the quotient $\left( k^{n+1} \setminus \{0\} \right) / k^\times$, where as usual $k^\times$ denotes $k \setminus \{0\}$: we identify a point in $\mathbb{P}^n(k)$ with any vector in the line in $k^{n+1}$ that is our point.

From the description in terms of a quotient above, we see that if we are to give coordinates to the points in $\mathbb{P}^n(k)$ we need $n + 1$ of them and they are not unique;
we can scale all coordinates by a non-zero element of \( k \). This matches with the identification of a point in \( \mathbb{P}^n \) and lines in \( k^{n+1} \) as for \( \lambda \in k^\times \) a vector \( v \in k^{n+1} \) and \( \lambda v \) span the same line.

2 Algebraic varieties

In algebraic geometry, we study objects that often can be described as subsets of \( \mathbb{P}^n \) in a specific way: as the zero set of one or more polynomial equations. Here we will restrict ourselves to those that are given by a single one.

Not any polynomial equation is allowed though, and this is to do with the non-uniqueness of coordinates. We only allow ‘homogeneous’ ones.

**Definition 2.1.** A *homogeneous polynomial* (in several variables) over a field \( k \) is one with coefficients in \( k \) where each of the monomials have the same total degree.

**Example 2.2.** Homogeneous polynomials in one variable are somewhat boring; they all look like \( \lambda X^d \) for some \( \lambda \in k \) and degree \( d \). In case of multiple variables, we get more interesting ones. For example we have \( X^2 + 5Y^2, XY - X^2, X^d + Y^d - Z^d \), etc.

The reason why we only allow homogeneous polynomials is the following: let \( \lambda \in k^\times \) be any scalar and \( f \in k[\{X_1, \ldots, X_n\}] \) (this is just notation for the ring of polynomials over \( k \) in \( n \) variables) any homogeneous polynomial of degree \( d \). Then we have \( f(\lambda x) = \lambda^d f(x) \). It is, for example, therefore not a good question to ask for all points \( x = (x_1, \ldots, x_n) \) such that \( f(x) = 1 \) holds, but it is a well-formed question to ask for such points \( x \) such that \( f(x) = 0 \) holds; here the scaling does not ruin anything.

**Fact 2.3.** A single homogeneous equation in \( n+1 \) variables gives an \( n-1 \)-dimensional algebraic variety in \( \mathbb{P}^n \). This does not depend on the degree of the polynomial equation.

**Remark 2.4.** From the last example above, one can see how algebraic geometry encompasses large parts of mathematics in general: solving Fermat’s famous equation for exponent \( d \) is equivalent to finding points on the curve \( X^d + Y^d - Z^d = 0 \) in \( \mathbb{P}^2 \) with coefficients in \( \mathbb{Q} \).

**Definition 2.5.** Let \( k \) be a field. Then we can consider \( \mathbb{P}^n(k) \) and a homogeneous equation \( f \in k[X_1, \ldots, X_{n+1}] \). The solutions of \( f(x) = 0 \) are called \( k \)-rational points of the algebraic variety given by \( f \).

3 Counting \( \mathbb{Q} \)-rational points

From now on, we take \( k = \mathbb{Q} \). If we are given any point \( (x_1, \ldots, x_{n+1}) \in \mathbb{P}^n(\mathbb{Q}) \), we can, by scaling, assume that for each \( i \), the coordinate \( x_i \) lies in \( \mathbb{Z} \) and moreover, that \( \gcd(x_1, \ldots, x_{n+1}) = 1 \) holds.

**Definition 3.1.** Let \( x = (x_1, \ldots, x_{n+1}) \) be a point in \( \mathbb{P}^n(\mathbb{Q}) \), scaled as above, then
the *height* of $x$ is

$$\text{ht}(x) = \max_{1 \leq i \leq n+1} |x_i|,$$

where, of course, $| \cdot |$ denotes the usual absolute value.

**Fact 3.2.** For any $B \in \mathbb{R}$ and for any algebraic variety $X \subset \mathbb{P}^n$, the number $N_X(B) := \#\{x \in X : \text{ht}(x) \leq B\}$ is finite.

If in the fact above, we let $B$ vary, we get a function $N_X : \mathbb{R} \to \mathbb{Z}$ given by $B \mapsto N_X(B)$. The growing behaviour of this function is known for some varieties, but in general still widely open to conjecture. An easy example is $X = \mathbb{P}^n$ itself.

**Proposition 3.3.** For $X = \mathbb{P}^n$, one has

$$N_X(B) \sim \frac{2^n}{\zeta(n+1)} B^{n+1}$$

as $B \to \infty$. Here $f(x) \sim g(x)$ means $f = g = 0$ or $\frac{f(x)}{g(x)} \to 1$ as $x \to \infty$ and $\zeta$ is the familiar Riemann zeta-function.

**Proof.** The proof of this is rather easy and only sketched here. In $\mathbb{P}^n$ each point carries $n+1$ coordinates, yielding the power $n+1$ for $B$. The Riemann zeta-function comes from the requirement that the coefficients be coprime, the power of 2 from choice of sign for each coordinate.

It is conjectured that for many, if not all varieties, the function $N_X(B)$ takes the form $N_X(B) \sim c B^a (\log B)^b$, where each of the constants $a, b, c$ have a (known) geometric interpretation.

There are counterexamples to these conjectures, but it might be possible to modify them somewhat. One known counterexample consist of a surface $X$ that has $a = 1$ from its geometric interpretation, but which also contains a line (i.e. a $\mathbb{P}^1$). As the points in $\mathbb{P}^1$ already grow as $B^2$, the conjecture cannot be true for such $X$. A suitable modification for this is not to take the whole $X$, but to consider an open subset where we have excluded a finite union of codimension 1 subvarieties (that is curves in case of a surface). More involved counterexamples are known that cannot be done away with by removing codimension 1 subvarieties and so far there is no consensus on how we should be counting what precisely.