An introduction to stable maps – Erik Visse

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These are the notes from the seminar on Moduli stacks of curves held in Leiden in the autumn of 2015. The website for this seminar can be found at `http://pub.math.leidenuniv.nl/~bommelrvan/2015/modstack.htm`.

These notes take heavily (!) after Kock and Vainsencher's explanatory text about Kontsevich' formula for rational plane curves [KV03].

The author thanks David Holmes for his help with the material and for his willingness to give the lecture introducing stable curves.

Without further mention we will only work over the complex numbers in this talk.

1 Maps $\mathbb{P}^1 \to \mathbb{P}^r$

We'll be treating maps from $\mathbb{P}^1$ to $\mathbb{P}^r$ for $r \geq 1$. In particular, in my next lecture will consider the case $r = 2$ and we'll count the number of rational curves of degree d in $\mathbb{P}^2$ going through any given $3d - 1$ points in general position. The present lecture will serve mostly as a set-up for that topic.

To tackle the problem mentioned above we will need so-called *stable maps*. In order to successfully apply the theory, we'll need a particular definition for the degree of a map that is slightly different from both the degree of a map between curves and the degree of a curve embedded in some projective space, in fact, our definition here combines the two.

Definition 1.1. Let C be a rational curve. The *degree* of a non-constant map $\mu\colon C \to \mathbb{P}^r$ is the degree of the image cycle $\mu_*[C]$. Phrased in an equivalent but more down-to-earth way, if d is the degree of the image curve C' and e is the degree of the map $C \to C'$ – i.e. the field extension degree $[\kappa(C') : \kappa(C)]$ – then the degree of μ is $d \cdot e$, again equivalently, it is the degree of $\mu^*\mathcal{O}_{\mathbb{P}^r}(1)$.

Remark 1.2. The degree of a constant map is 0.

We'll introduce the adjective *stable* for such maps in the next section and we'll consider the moduli space for such maps of given degree. In general when studying moduli spaces, if some objects have only trivial automorphisms one may expect a fine moduli space whereas if such objects have finitely many automorphisms, one may expect only a coarse moduli space. We'll encounter both situations later on, but let's first recall isomorphisms of maps for the sake of completeness.

Definition 1.3. An *isomorphism of maps* $\mu\colon C \to \mathbb{P}^r$ and $\mu'\colon C' \to \mathbb{P}^r$ is an isomorphism $\phi\colon C \xrightarrow{\sim} C'$ satisfying $\mu = \mu' \circ \phi$.

Lemma 1.4. *Let $\mu\colon \mathbb{P}^1 \to \mathbb{P}^r$ be a non-constant map, then there exist only finitely many automorphisms of μ. If furthermore μ is birational onto its image then* $\mathrm{Aut}(\mu)$ *is trivial and vice-versa.*

Proof. is Let C be the image curve of μ. Then the automorphism group of μ is naturally isomorphic to the group of automorphisms of $\kappa(\mathbb{P}^1)$ fixing $\kappa(C)$ pointwise, which is well known to be finite. The birationality condition is nothing else then $\kappa(C) = \kappa(\mathbb{P}^1)$. □

2 Stable maps

For today, we'll use the following definition that takes after definitions from both Giulio's lecture and David's lecture.

Definition 2.1. A *tree of rational curves* is a connected curve with the following properties:

(1) there are at worst nodal singularities,

(2) each irreducible component is a rational curve,

(3) the number of irreducible components is one more than the number of nodes.

We call the irreducible components *twigs*.

Definition 2.2. An *n-pointed map* is a morphism $\mu\colon C \to \mathbb{P}^r$ where $\pi\colon C \to \operatorname{Spec}\mathbb{C}$ is a tree of rational curves with n (distinct) marked smooth points, i.e. a map with n sections $\sigma_1, \ldots, \sigma_n$ of π such that the images of the σ_i's are disjoint and are all smooth points.
An *isomorphism* of n-pointed maps $\mu\colon C \to \mathbb{P}^r$ and $\mu'\colon C' \to \mathbb{P}^r$ is an isomorphism $\phi\colon C \to C'$ of maps further satisfying $\pi = \pi' \circ \phi$ and $\phi \circ \sigma_i = \sigma_i'$ for all $1 \leq i \leq n$.
We'll use the notation $(C; p_1, \ldots, p_n; \mu)$ for an n-pointed map $\mu\colon C \to \mathbb{P}^r$ together with its n marked points $p_1, \ldots, p_n$.

A family of n-pointed maps (and their isomorphisms) are easily defined as a map from a family of n-pointed curves $\mathfrak{X}$ to $\mathbb{P}^r$ and further requiring that this map restricted to any fibre is an n-pointed map as above.

We now arrive at the main definition for today.

Definition 2.3. An n-pointed map $\mu\colon C \to \mathbb{P}^r$ is called *(Kontsevich) stable* if any twig that is mapped to a point is a rational stable pointed curve.

Recall that a rational stable pointed curve has at least 3 special points on it, where a special point is either a marked or a singular point.

Remark 2.4. The domain of a stable map needn't be a stable curve: no requirements are made of any twig that isn't mapped to a point, in particular making any non-constant map $\mathbb{P}^1 \to \mathbb{P}^r$ a stable map.

Remark 2.5. Last week, David has told us that the automorphism group of a stable curve is always finite. If we only consider rational stable curves, then the automorphism group is in fact trivial as each rational curve is assumed to have at least 3 special points.

Lemma 2.6. *An n-pointed map is stable if and only if it has a finite automorphism group.*

Proof. Let $(C; p_1, \dots, p_n; \mu)$ be a stable map. If $(C; p_1, \dots, p_n)$ is a rational stable curve, then the automorphism group is trivial. If there exists a twig E that is unstable as a pointed curve, then μ is non-constant (by stability of μ). Let ϕ be any automorphism of μ and write $E' = \phi(E)$ for the image curve of E. Then $\mu|_{E'} \circ \phi|_E = \mu|_E$ holds, so Lemma 1.4 yields that there are only finitely many such $\phi|_E$. Since a stable curve has only finitely many twigs, we arrive at the desired result.
Conversely, suppose that $(C; p_1, \dots, p_n; \mu)$ is not stable. Then there is a twig E of the source curve C that is not a stable curve, yet is mapped to a single point. Then E has infinitely many automorphisms that all extend to an automorphism of C that is the identity away from E. Since $\mu(E)$ is a point, each of these automorphisms commute with μ. □

Theorem/Definition 2.7. *There exists a coarse moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ of isomorphism classes of stable n-pointed maps of degree d and a fine moduli space $\overline{M}^*_{0,n}(\mathbb{P}^r, d)$ of such maps without non-trivial automorphisms.*

Theorem 2.8. *$\overline{M}_{0,n}(\mathbb{P}^r, d)$ is a projective normal irreducible variety and it is locally isomorphic to a quotient of a smooth variety by the action of a finite group. It contains $\overline{M}^*_{0,n}(\mathbb{P}^r, d)$ as a smooth open dense subvariety.*

Lemma 2.9. *The dimension of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is $rd + r + d + n - 3$.*

Proof. Omitted. □

Bibliography

[KV03] Joachim Kock and Israel Vainsencher. Kontsevich's formula for rational plane curves. available on `http://www.dmat.ufpe.br/~israel/kontsevich.html`, 2003.