Examples of K3 surfaces and an introduction to Hodge decomposition – Erik Visse

These are the notes from the seminar on K3 surfaces and their automorphisms held in Leiden at the end of 2014. The website for this seminar can be found at `http://pub.math.leidenuniv.nl/~vissehd/k3seminar/`.

Definition 0.1. A *variety* over a field k is a separated, geometrically integral scheme of finite type.

1 Definition and examples of K3 surfaces

Definition 1.1. A *K3 surface* X over a field k is a complete, smooth and projective variety of dimension 2 such that

- the canonical sheaf is trivial; $\omega_X \cong \mathcal{O}_X$, and
- also the cohomology group $H^1(X, \mathcal{O}_X)$ is trivial.

We start of with some remarks:

Remark 1.2. The name K3 surface comes from Weil, who named them in honour of Kummer, Kähler and Kodaira, as well as the mountain named K2 in the Karakoram mountain range.
Surfaces are usually classified via their Kodaira dimension, which is defined below. In Kodaira dimension 0, we find not only K3 surfaces but also others with trivial canonical sheaf, for example abelian surfaces. The second requirement rules these out. We will however encounter abelian surfaces in giving examples of K3s.
We will often switch points of view between invertible sheafs and divisors. We denote the canonical divisor with K_X or K. In this language the triviality of the canonical divisor reads $K \sim 0$ with $\sim$ denoting linear equivalence of divisors.
In the seminar, we will usually work with K3 surfaces over $k = \mathbb{C}$.

Definition 1.3. Let L be an invertible sheaf on a smooth variety X. Let $N(L) = \{m \in \mathbb{Z}_{\geq 0} : H^0(X, L^m) \neq 0\}$. For $m \in N(L)$, we have a birational map

$$\phi_m : X \dashrightarrow \mathbb{P}H^0(X, L^m).$$

The *Kodaira dimension* of X is

$$\kappa(X) = \sup_{m \in N(K_X)} \{\dim \phi_m(X)\}$$

if $N(K_X)$ is non-empty and $\kappa(X) = -\infty$ if it is.

Fact 1.4. A complete and smooth surface is always projective. We therefore could have dropped this requirement from the definition, but we left it in for easy reference.

We now come to the first and most basic example of a K3 surface.

Example 1.5. *A smooth quartic in* $\mathbb{P}^3$. Let $i : X \hookrightarrow \mathbb{P}^3$ be given by a smooth quartic. Then X is a K3 surface.

Proof. The ideal sheaf of X is $\mathcal{O}_{\mathbb{P}^3}(-4)$ and fits in an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_X \to 0,$$

where we are (and will continue to be) sloppy in notation and write $\mathcal{O}_X$ instead of $i_*\mathcal{O}_X$.

From the accompanying exact sequence in cohomology we get

$$\ldots \to H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}) \to H^1(\mathbb{P}^3, \mathcal{O}_X) \to H^2(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) \to \ldots$$

and we use the following fact from [Har77]Theorem III 5.1: for $0 \neq i \neq n$ and for all $d \in \mathbb{Z}$, we have $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = 0$.

So we get $H^1(\mathbb{P}^3, \mathcal{O}_X) = 0$ and find the second requirement for being a K3 surface from the isomorphism $H^1(\mathbb{P}^3, \mathcal{O}_X) \cong H^1(X, \mathcal{O}_X)$.

To compute the canonical sheaf of X, we use the formula from [Har77]Proposition 8.20: if $\mathcal{L}(X)$ is the sheaf associated to $X \subset \mathbb{P}^3$ seen as a divisor, we have

$$\omega_X \cong \omega_{\mathbb{P}^3} \otimes \mathcal{L}(X) \otimes \mathcal{O}_X.$$

As $\mathcal{L}(X) \cong \mathcal{O}_{\mathbb{P}^3}(4)$ holds and $\omega_{\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^3}(-4)$, we conclude $\omega_X \cong \mathcal{O}_X$. $\square$

Example 1.6. This example can easily be generalized to complete intersections X in $\mathbb{P}^{n+2}$ of type $(d_1, d_2, \ldots, d_n)$. Here we have $\omega_{\mathbb{P}^{n+2}} \cong \mathcal{O}_{\mathbb{P}^{n+2}}(-n-3)$ and

$$\mathcal{L}(X) \cong \mathcal{O}_{\mathbb{P}^{n+2}}(d_1 + \ldots + d_n).$$

Furthermore, we have $H^1(X, \mathcal{O}_X) = 0$ from induction to the number of defining hypersurfaces and what we have seen in Example 6.

So we get a K3 surface if and only if $\sum_i d_i = n + 3$ holds. If we don't want X to actually live in a lower dimensional $\mathbb{P}^N$, we need all the d_i's to be at least 2 and get $2n \leq \sum_i d_i = n + 3$ and thus $n \leq 3$. We have (up to ordering) only three cases:

- $n = 1$ which we have already treated,
- $n = 2$, $d_1 = 2, d_2 = 3$,
- $n = 3$, $d_1 = d_2 = d_3 = 2$.

In fact, there exist many more K3 surfaces than only those we have seen above, but they are necessarily more complicated.

We will treat one more in some detail and only mention another.

Example 1.7. Let A be an abelian surface. There exists an involution $\tau : A \to A$ sending a point p to $-p$. The quotient $A/\langle\tau\rangle$ has 16 singular points, namely the (images of the) points of order 2. Therefore $A/\langle\tau/\rangle$ cannot be a $K3$ surface. Its minimal resolution is.
A different way to accomplish the same thing, is first to blow up A in the sixteen points of order 2 and then take the corresponding quotient. Let $\varepsilon : \tilde{A} \to A$ be this blow-up. Then τ induces an involution σ on $\tilde{A}$ by $\sigma(p) = p$ for any point p in one of the exceptional divisors and by $\sigma(p) = \tau(p)$ for p outside the union of the exceptional divisors. Then $X = \tilde{A}/\langle\sigma\rangle$ is a K3 surface.

Proof. To see that X is complete, we notice that $\tilde{A}$ is projective and that it is a double cover of X.

We only need to show smoothness of X at the exceptional curves since the quotient map $\pi : \tilde{A} \to X$ is étale. This can be shown by a local calculation, which in can be found in [Bea96] Proposition VIII.11 and which will not be repeated here.

We will first show $K_X \sim 0$. We start by noting that from an isomorphism $A \cong \mathbb{C}^2/\Lambda^2$ and x, y coordinates on $\mathbb{C}^2$, we get a nowhere zero holomorphic 2-form $\omega = dx \wedge dy$ on A. In particular, ω is invariant under τ since the signs of both x and y are switched by τ. Therefore also $\varepsilon^*\omega$ is invariant under σ. By Lemma VI.11 from [Bea96], there exists a meromorphic 2-form on X such that $\varepsilon^*\omega = \pi^*\alpha$ holds. The support of the divisor of α must be contained in the union of the exceptional curves. Another local computation shows that α is in fact holomorphic and non-zero there too, so $K_X = \mathrm{div}(\alpha) = 0$ holds.

To proof $H^1(X, \mathcal{O}_X) = 0$, it suffices to show $H^0(X, \Omega^1_X) = 0$ by the existence of a weight 1 Hodge decomposition (which itself is to be treated later, and the existence of which must be shown separately and is not done here). Here Ω^1_X is the sheaf of differential 1-forms. Let $\alpha \in H^0(X, \Omega^1_X)$ be non-zero. Then $\pi^*\alpha = \beta \in H^0(\tilde{A}, \Omega^1_{\tilde{A}})$ is non-zero and invariant under σ. Since ε is birational, from the proof of Proposition III.20 in [Bea96], we find that $\varepsilon^* : H^0(A, \Omega^1_A) \to H^0(\tilde{A}, \Omega^1_{\tilde{A}})$ is an isomorphism such that also $(\varepsilon^*)^{-1}\beta = \gamma$ is invariant under τ. But since $H^0(A, \Omega^1_A)$ has basis $\{dx, dy\}$, both of which are not invariant under τ, this cannot happen. □

This X is called the *Kummer surface* associated to A. A different proof can be found in [Bad01].

Example 1.8. The last example of a K3 surface which we mention is a double cover $\pi : X \to \mathbb{P}^2$ branched along a degree 6 smooth curve.

2 Hodge theory

We'll now introduce some of the theory that will be important later on. This will involve a stroll into some differential geometry. This part is somewhat of a sketch only.

Let M be a compact complex manifold. We need some notation, and leave details to be found in other texts, such as [BHPV04].

Definition 2.1. We write $\mathcal{D}^p_M$ for the sheaf of real-valued C^∞ p-forms on M and $H^p_{DR}(M)$ for the p-th de Rham group

$$H^p_{DR}(M) = \{\alpha \in \mathcal{D}^p_M(M) | d\alpha = 0\}/d\mathcal{D}^{p-1}_M(M),$$

that is, the group of closed p-forms modulo exact p-forms.

It is a general fact that the p-th de Rham group is isomorphic to the p-th singular cohomology group (that we will not go into here) $H^p(M, \mathbb{Z}) \otimes \mathbb{R} = H^p(M, \mathbb{R})$.

Definition 2.2. We further write $H^p(M,\mathbb{C}) = H^p(M,\mathbb{R}) \otimes \mathbb{C}$. The (complex) dimension of $H^p(M,\mathbb{C})$ is denoted b_p and called the p-th Betti number.

Definition 2.3. Let V either a free $\mathbb{Z}$-module of finite rank, a $\mathbb{Q}$-vector space of finite dimension or an $\mathbb{R}$-vector space of finite dimension. A Hodge structure of weight $n \in \mathbb{Z}$ on V is a direct sum decomposition of $V \otimes \mathbb{C} = V_{\mathbb{C}}$

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$$

where $V^{p,q} = \overline{V^{q,p}}$ holds. Here, conjugation is applied on the right-hand factor $\mathbb{C}$ in the tensor product.

Definition 2.4. We write $\mathcal{D}_M^{p,q}$ for the sheaf of $\mathbb{C}$-valued C^∞-forms of type (p,q). That is, forms that in local coordinates $z_1, \ldots, z_n$ are linear combinations of

$$dz_{i_1} \wedge \ldots \wedge dz_{i_p} \wedge d\overline{z_{j_1}} \wedge \ldots \wedge d\overline{z_{j_q}}.$$

We further write

$$H^{p,q}(M) = \{\alpha \in \mathcal{D}_M^{p,q}(M) | \overline{\partial}\alpha = 0\} / \overline{\partial}\mathcal{D}_M^{p,q-1}(M).$$

Theorem 2.5. *Let X be a smooth, quasi-compact surface over $\mathbb{C}$. Then $H^2(X,\mathbb{Z})$ has a Hodge structure*

$$H^2(X,\mathbb{C}) = \bigoplus_{p+q=2} H^{p,q}(X)$$

and $H^1(X,\mathbb{Z})$ has a Hodge structure if and only if $b_1(X)$ is even.

Theorem 2.6. *For each non-negative p,q, there is an isomorphism*

$$H^{p,q}(M) \cong H^q(M, \Omega_M^p).$$

Theorem 2.7. *If X is a K3 surface, then $b_0 = 1$, $b_1 = 0$, $b_2 = 22$, $b_3 = 0$ and $b_4 = 1$.*

Remark 2.8. For a complex surface X the groups $H^i(X,\mathbb{Z})$ for $i > 4$ are trivial since X has real dimension 4.

Example 2.9. Consider $M = \mathbb{C}^2 \setminus \{0\}$ and let the discrete group $2\mathbb{Z}$ act on M by multiplication on both coordinates. Then $M/2\mathbb{Z}$ is a compact complex surface with $b_1 = 1$. Surfaces of this form are called Hopf surfaces.

References

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[Har77] Robin Hartshorne, *Algebraic Geometry*, Springer Science+Business Media, Inc., New York, 1977