Lecture 11 – Erik Visse

These are the notes from the seminar on K3 surfaces and their automorphisms held in Leiden at the end of 2014 and start of 2015. The website for this seminar can be found at `http://pub.math.leidenuniv.nl/~vissehd/k3seminar/`.

As always, K3 surfaces are assumed to be projective. If the base field is unmentioned, it is assumed to be the field of complex numbers.

1 Hodge isometries of $\mathrm{T}(X)$ in odd Picard number

We start by recalling the definition of the transcendental lattice without reference to the Néron–Severi lattice.

Definition 1.1. Let V be a free $\mathbb{Z}$-module of finite rank and let n be an integer. A *Hodge structure of weight* n on V is a direct sum decomposition of $V_{\mathbb{C}}$ as

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$$

with $V^{p,q} = \overline{V^{q,p}}$ where complex conjugation acts on the factor $\mathbb{C}$ in $V_{\mathbb{C}} = V \otimes \mathbb{C}$.

Definition 1.2. A *sub-Hodge structure of weight* n is given by a sub-module $V' \subset V$ such that the Hodge structure on V induces a Hodge structure on V' by $(V')^{p,q} = V'_{\mathbb{C}} \cap V^{p,q}$. A sub-Hodge structure is called *primitive* if V/V' is torsion free.

Definition 1.3. A *Hodge structure of K3 type* is a Hodge structure V of weight 2 with $V^{p,q} = 0$ for $|p-q| > 2$ and $\dim_{\mathbb{C}} V^{2,0} = 1$.

Definition 1.4. Let V be a Hodge structure of K3 type. The transcendental part is the minimal primitive sub-Hodge structure $T \subset V$ such that $V^{2,0} = T^{2,0} \subset T_{\mathbb{C}}$ holds.

Indeed, for a K3 surface X, the cohomology group $H^2(X, \mathbb{Z})$ has a well-known Hodge structure which is of K3 type and the transcendental lattice – which is the orthogonal complement of $\mathrm{NS}(X) = H^2(X,\mathbb{Z}) \cap H^{1,1}$ inside $H^2(X,\mathbb{Z})$ – is the transcendental part, see for example [Huy14, Lem. 3.4.1].

Proposition 1.5 (Huy14, Cor 3.4.5). *Let X be a K3 surface with odd Picard number. Then the only Hodge isometries of $T(X)$ are $\{\pm 1\}$.*

Proof. Let ϕ be a Hodge isometry of $T(X)$. Then ϕ has eigenvalue λ if and only if $\overline{\lambda} = \lambda^{-1}$ is also an eigenvalue. If $\rho(X)$ is odd, then $\lambda = \pm 1$ must occur since the non-real eigenvalues come in pairs.

Let $0 \neq \alpha \in T(X) \subset H^2(X,\mathbb{Z})$ be an eigenvector of ϕ with eigenvalue ± 1. If necessary composing by $-\mathrm{id}$, we may assume that this eigenvalue is 1. Now assume that $\phi \neq \mathrm{id}$. As we have seen before, there is a root of unity ζ such that $\phi = \zeta\,\mathrm{id}$ on $H^{2,0}(X)$. We have $\zeta \neq 1$ since a Hodge isometry that is the identity on $H^{2,0}$ is the identity on $T(X)_{\mathbb{C}}$. Let $\beta \in H^{2,0}(X)$ be arbitrary. Then we have

$$(\alpha, \beta) = (\phi(\alpha), \phi(\beta)) = (\alpha, \zeta\beta)$$

and therefore $(\alpha, (1-\zeta)\beta) = 0$. As $H^{2,0}(X)$ is 1-dimensional and the pairing on $H^2(X, \mathbb{C})$ is non-degenerate, we conclude $\alpha \in (H^{2,0})^{\perp}$.

By assumption the projection of α on $H^{2,0}$ is 0 (again since $H^{2,0}$ is 1-dimensional, so we get $\alpha \in H^{1,1}$. But we already know $H^{1,1} \cap H^2(X, \mathbb{Z}) = \mathrm{NS(X)}$, contradicting $\alpha \neq 0$. $\square$

Remark 1.6. The above proposition does not guarantee that $-\mathrm{id}$ actually occurs as coming from an automorphism.

Using this proposition, we can prove the result I only mentioned in my last lecture, which classifies the automorphism groups of K3 surfaces with Picard number 1.

Proposition 1.7 (Huy14, Cor 15.2.12). *Let X be a K3 surface with* $\mathrm{Pic}(X) \cong \mathbb{Z}\cdot H$.

- *If $H^2 > 2$, then* $\mathrm{Aut}(X) = \{\mathrm{id}\}$,
- *if $H^2 = 2$, then* $\mathrm{Aut}(X) \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. We can apply the previous proposition, so every $f \in \mathrm{Aut}(X)$ will act as $\pm\,\mathrm{id}$ on $T(X)$. At the same time, any f will act as id on $\mathrm{NS}(X)$ since H is mapped to H. Furthermore the induced actions on $A_{T(X)} = A_{\mathrm{NS}(X)} \cong \mathbb{Z}/H^2\mathbb{Z}$ must agree. For $H^2 > 2$ we conclude $f^* = \mathrm{id}$ and by the Torelli theorem also $f = \mathrm{id}$.
If however $H^2 = 2$ holds, then X is a double plane and the involution i coming from the double cover acts as $-\,\mathrm{id}$ on $T(X)$. Suppose that $f \in \mathrm{Aut}(X)$ also acts as $-\,\mathrm{id}$ on $T(X)$, then we have

$$(f \circ i)* = f^* \circ i^* = -\,\mathrm{id} \circ -\,\mathrm{id} = \mathrm{id}$$

and therefore $f \circ i = \mathrm{id} \in \mathrm{Aut}(X)$ and thus $f = i^{-1} = i$. $\square$

2 An example in rank 3

Let S be a K3 surfaces that is a double cover of the plane $f : S \to \mathbb{P}^2$ ramified over a sextic curve C and assume that C is smooth. Furthermore assume that C has two tritangents ℓ_1 and ℓ_2. Since ℓ_i is ample, f is finite, and $\mathbb{P}^2$ and S are complete, $f^*\ell_i =: L_i$ is ample (Laz04, Prop. 1.2.23). As furthermore ℓ_1 and ℓ_2 are rationally equivalent, so are L_1 and L_2. Denote their class by L. We have

$$L^2 = L_1 \cdot L_2 = f^*\ell_1 \cdot f^*\ell_2 = (\deg f)\ell_1 \cdot \ell_2 = 2.$$

The divisor L_1 decomposes as $L_1 = N_1 + N_2$ and L_2 decomposes as $L_2 = N_3 + N_4$. Since ℓ_1 and ℓ_2 are tritangents, we have $N_1 \cdot N_2 = 3 = N_3 \cdot N_4$. Expanding the square L^2 and using that the self-intersection of a curve on a K3 surface is at least -2, we find for $i = 1, \dots, 4$ the self-intersection $N_i^2 = -2$.

The assumption that C be smooth allows us to assume without loss of generality that N_2 and N_3 do not intersect, but N_1 and N_3 respectively N_2 and N_4 do intersect (with multiplicity 1).

With respect to the basis L, N_2, N_3, we get an intersection matrix

$$J = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix}.$$

PROPOSITION 2.1. *Assuming $\rho(S) = 3$, the Néron–Severi group of S is spanned by L, N_2, N_3.*

Proof. The discriminant of the lattice spanned by L, N_2, N_3 equals the determinant of J, which is 12. The assumption $\rho(S) = 3$ implies that $\langle L, N_2, N_3 \rangle$ is of finite index m inside $\mathrm{NS}(S)$ and m^2 divides $\det(J) = 12$. Assume towards a contradiction that $m = 2$ holds.
Then for every divisor (class) $D \in \mathrm{NS}(S)$ we can write $D = aL + bN_2 + cN_3$ with $a, b, c \in \mathbb{Z}[\frac{1}{2}]$. Suppose that such a D is given. Then we calculate

$$D^2 = 2a^2 - 2b^2 - 2c^2 + 2ab + 2ac; \tag{0}$$
$$D \cdot L = 2a + b + c; \tag{1}$$
$$D \cdot N_2 = a - 2b; \tag{2}$$
$$D \cdot N_3 = a - 2c. \tag{3}$$

By either (2) or (3) we know $a \in \mathbb{Z}$ and by (1) we know $b + c \in \mathbb{Z}$. Since S is a K3 surface, we know that self-intersections are even integers, so from (0) we know $a^2 - b^2 - c^2 + a(b + c) \in \mathbb{Z}$ and therefore $b^2 + c^2 \in \mathbb{Z}$. Writing $b = b'/2$ and $c = c'/2$, we get $(b')^2, (c')^2 \in 4\mathbb{Z}$ and reducing modulo 4, we find $b', c' \in 2\mathbb{Z}$ so $b, c \in \mathbb{Z}$. This contradicts $m = 2$ and we conclude $m = 1$ and therefore that $\langle L, N_2, N_3 \rangle = \mathrm{NS}(S)$. □

2.1 COMPUTING THE AMPLE CONE

From now on, we will keep the assumption $\rho(S) = 3$, i.e. 'S is general enough'.

DEFINITION 2.2. We use the notation $\mathcal{L}^+ = \{x \in \mathrm{NS}(S) | x^2 > 0 \text{ and } x \cdot H > 0\}$ where H is any ample divisor and $\mathcal{L} = \{x \in \mathrm{NS}(S) | x^2 > 0\}$. We have previously called $\mathcal{L}^+$ the 'positive cone', but another term that is used is the 'light cone', originating in its hyperbolic nature, also encountered in physics in relativity theory.

LEMMA 2.3. *Let X be any K3 surface and let $\mathcal{N}$ be a set of (-2)-curves on X. Let $\mathcal{C}_\mathcal{N}$ be the cone bounded by walls $N^\perp$ for all $N \in \mathcal{N}$*[1]*. If $\mathcal{C}_\mathcal{N} \subset \mathcal{L}$ holds, then $\mathcal{N}$ is the set of all $(-2)-curves$.*

Proof. Suppose that there is a (-2)-curve $N \notin \mathcal{N}$. For all $N' \in \mathcal{N}$ we have $N \cdot N' \geq 0$, thus by assumption $N \in \mathcal{C}_\mathcal{N} \subset \mathcal{L}$ and therefore $N^2 > 0$, contradicting N to be a (-2)-curve. □

PROPOSITION 2.4. *The curves $N_1, \ldots, N_4$ are the only (-2)-curves on S.*

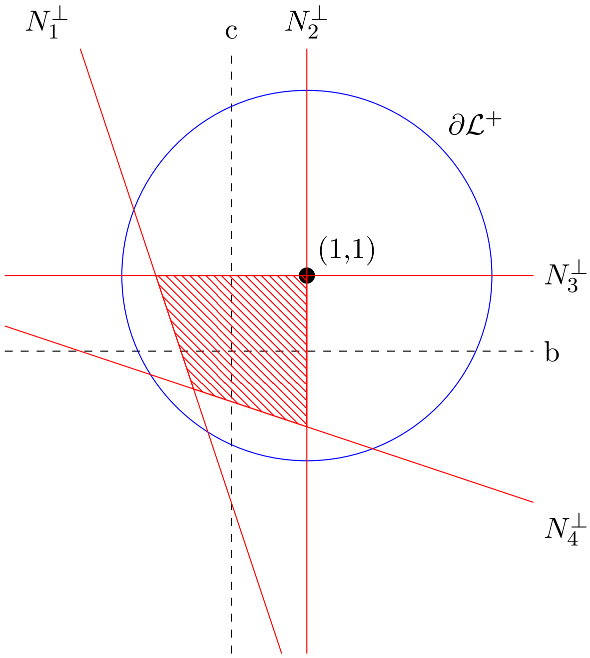
[1] i.e. $\mathcal{C}_\mathcal{N} = \{x \in \mathrm{NS}(S) | \forall N \in \mathcal{N} : x \cdot N \geq 0\}$

Proof. A quick computation shows

$$\mathcal{L}^+ = \left\{ aL + bN_2 + cN_3 \in \mathrm{NS}(S) \,\middle|\, \begin{array}{c} \left(b - \frac{a}{2}\right)^2 + \left(c - \frac{a}{2}\right)^2 < \frac{3}{2}a^2 \\ a > 0 \end{array} \right\}$$

where $a > 0$ is found by remarking that L is ample.

If we now take the cross-section at $a = 2$, we get the pictorial presentation below and we conclude using the lemma.



□

Since the ample cone is bounded by walls corresponding to all (-2)-classes (and by $\mathcal{L}^+$, we have found the ample cone of S.

2.2 Computing $O(\mathrm{NS}(S))$ and $O(T(S))$

We know that $L \cdot N_i = 1$ holds for $i = 1, \dots, 4$. Let $\phi \in \mathrm{Aut}(S)$ be arbitrary, then $\phi^*\, \mathrm{O}(\mathrm{NS}(S))$ permutes the (-2)-curves $N_1, \dots, N_4$ and preserves intersections. Therefore we get $\phi^*(L) \cdot N_i = 1$ for $i = 1, \dots, 4$. Writing $\phi^*(L) = aL + bN_2 + cN_3$, we find by considering all intersections $a = 1$ and $b = c = 0$. Therefore ϕ^* fixes L. To find the action of ϕ^* on $\mathrm{NS}(S)$, we only need consider $\phi^*(N_2)$ and $\phi^*(N_3)$ and we already know that these must be some N_i and N_j respectively.

Proposition 2.5. *The group $O(\mathrm{NS}(S))$ consists of the following four elements*

$$\mathrm{id}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} =: P, \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = i^*, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} = Pi^* = i^*P.$$

where $i : S \to S$ is the involution provided by the double cover.

Proof. This can be found by quickly considering all possibilities for the images of N_2 and N_3. $\square$

PROPOSITION 2.6. *We have* $O(T(S)) = \{\pm \mathrm{id}\}$.

Proof. This is just Proposition 1.5 combined with the fact that i induces $-\mathrm{id}$ on $T(S)$. $\square$

2.3 COMPUTING A_{NS}

The discriminant lattice A_{NS} has order $\det(J) = 12$ and is abelian so it can only be (isomorphic to) two possible groups: C_{12} and $C_6 \times C_2$. We can find generators for it by inverting J. We have

$$J^{-1} = \frac{1}{12}\begin{pmatrix} 4 & 2 & 2 \\ 2 & -5 & 1 \\ 2 & 1 & -5 \end{pmatrix}$$

and A_{NS} is generated by

- $b_1 := \frac{1}{3}L + \frac{1}{6}N_2 + \frac{1}{6}N_3$,
- $b_2 := \frac{1}{6}L + \frac{-5}{12}N_2 + \frac{1}{12}N_3 \equiv \frac{1}{6}L + \frac{5}{12}N_2 + \frac{1}{12}N_3$, and
- $b_3 :\equiv \frac{1}{6}L + \frac{1}{12}N_2 + \frac{7}{12}N_3$.

In fact, we have $2b_2 = b_1$ and $7b_2 = b_3$, so A_{NS} is cyclic of order 12.

PROPOSITION 2.7 (Huy14, Thm 14.2.4). *Let ℓ denote the minimum number of generators for A_{NS}. If $\ell + 2 \leq \rho$ holds, then the natural map $O(\mathrm{NS}) \to O(A_{\mathrm{NS}})$ is surjective.*

This proposition is actually a general lattice theoretic statement where the lattice is only required to be even and indefinite. The premise of this proposition is satisfied and in fact, both groups are isomorphic to the Klein four group.

2.4 COMPUTING $\mathrm{Aut}(S)$

We know that $\mathrm{Aut}(S) \cong \{(\alpha, \beta) | \overline{\alpha} = \overline{\beta}\} \subset O(\mathrm{NS}(S)) \times O(T(S))$ holds, where $\overline{\phi}$ means the map that ϕ induces on $A_{\mathrm{NS}} = A_{T(S)}$. Thus to find $\mathrm{Aut}(S)$, we need to consider what the maps found in Proposition 2.5 induce on A_{NS}. We find

- $\overline{\mathrm{id}} b_2 = b_2$,
- $\overline{i^*} b_2 = -b_2$,
- $\overline{P} b_2 = b_3 \neq \pm b_2$, and
- $olPi^* b_2 = -b_3 \neq \pm b_2$

so only id and i^* can come from an automorphism of the surface, and in fact they trivially do. We have found:

THEOREM 2.8. *The automorphism group of S is* $\mathrm{Aut}(S) = \{\mathrm{id}, i\}$.

References

[Huy14] Daniel Huybrechts, *Lecture notes on K3 surfaces*, 2014, available at `http://www.math.uni-bonn.de/people/huybrech/K3Global.pdf`, last accessed on January 13th, 2015

[Laz04] Robert Lazarsfeld, *Positivity in Algebraic Geometry I*, Springer-Verlag, Berlin Heidelberg, 2004