An example by Baragar in rank 4 – Erik Visse

These are the notes from the seminar on K3 surfaces and their automorphisms held in Leiden at the end of 2014 and start of 2015. The website for this seminar can be found at `http://pub.math.leidenuniv.nl/~vissehd/k3seminar/`.

As always, K3 surfaces are assumed to be projective. If the base field is unmentioned, it is assumed to be the field of complex numbers.

1 An example by Baragar in rank 4

In this section, we consider the paper 'The ample cone for a K3 surface written by Arthur Baragar [Bar11]. We will need quite a bit of hyperbolic geometry, and moreover some geometric imagination to interpret the pictures coming up in the paper. I hope to be able to properly explain them during the seminar, but I fear I will not be quite able conveying their meaning on paper.

We fix some notation. We will study a (class of) K3 surface(s) over a number field which we will denote by V. With respect to some basis of $\mathrm{NS}(V)$ to be chosen, we will write J for the intersection matrix. We write $\mathcal{O}$ for the group of isometries of $\mathrm{NS}(V)$, i.e. those represented by integer matrices that preserve J. Given a ample divisor H, there is a hypersurface in $\mathrm{NS}(V)$ determined by the equation $xJx = H^2$ which is hyperbolic and contains two sheets. We distinguish the one that contains H and denote it $\mathcal{H}$. The subgroup of $\mathcal{O}$ that preserves $\mathcal{H}$ is denoted $\mathcal{O}^+$. The subgroup of $\mathcal{O}^+$ that preserves the ample cone is denoted $\mathcal{O}''$.
We write $\mathcal{L} = \{x \in \mathrm{NS}(V) | x^2 > 0\}$ and $\mathcal{L}^+ = \{x \in \mathrm{NS}(V) | x^2 > 0, x \cdot H > 0\}$.

Proposition 1.1. *A smooth surface given by a $(2,2,2)$-form in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is a K3 surface.*

Proof. Let X denote such a surface. We mimic the proof of that of a smooth quartic in $\mathbb{P}^3$. The canonical sheaf of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}(-2,-2,-2)$. We have

$$\omega_X \cong \omega_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1} \otimes \mathcal{O}(X) \otimes \mathcal{O}_X$$

and further have $\mathcal{O}(X) = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}(2,2,2)$ so we directly conclude $\omega_X \cong \mathcal{O}_X$.

We use the exact sequence in cohomology associated to the embedding sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1}(-2,-2,-2) \to \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1} \to \mathcal{O}_X \to 0$$

together with Serre duality for $H^2(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-2,-2,-2))$ to reduce the proof of triviality of $H^1(X, \mathcal{O}_X)$ to proving the triviality of $H^1(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1})$. Since $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is birational to $\mathbb{P}^3$, it follows from the fact that the dimension of $H^i(Y, \mathcal{O}_Y)$ is a birational invariant for all i and all smooth proper schemes Y over a field of characteristic zero. □

A $(2,2,2)$-from in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is the zero locus of a polynomial of the form

$$F(X,Y,Z) = X_0^2 F_0(Y,Z) + X_0 X_1 F_1(Y,Z) + X_1^2 F_2(Y,Z) \tag{1}$$

where $X = (X_0 : X_1)$ and the $F_i(Y,Z)$ are $(2,2)$-forms in $\mathbb{P}^1 \times \mathbb{P}^1$.

PROPOSITION 1.2. *A smooth* $(2,2)$*-form in* $\mathbb{P}^1 \times \mathbb{P}^1$ *is an elliptic curve.*

Proof. Let $E : 0 = G(Y,Z) = Y_0^2 G_0(Z) + Y_0 Y_1 G_1(Z) + Y_1^2 G_2(Z)$ be a smooth $(2,2)$-form in $\mathbb{P}^1 \times \mathbb{P}^1$. There is a natural $2:1$ cover $\pi : E \to \mathbb{P}^1$ by $(Y,Z) \mapsto Z$. Invoking the Hurwitz formula for branched coverings, we only need to show that π is ramified along 4 points. □

Let V be the K3 surface defined by the smooth $(2,2,2)$-form $F(X,Y,Z)$, on which for now we don't impose any additional restrictions. There is a projection $p_1 : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ given by $(X,Y,Z) \mapsto X$ and similarly there are projections p_2 and p_3. Fixing a point $P \in \mathbb{P}^1$, we fix divisor classes $D_i = V \cap p_i^{-1} P$ on V. Then D_1, D_2 and D_3 are linearly independent in $\mathrm{NS}(V)$. Their mutual intersection numbers are given byAn example by Baragar in rank 4

$$D_i \cdot D_j = 2\,(1 - \delta_{ij})\,.$$

Viewing equation (1) as a quadric equation in X with parameters Y, Z, then for each Y and Z there are two solutions X and X' and we get a map $\sigma_1 : V \to V$ given by $(X,Y,Z) \mapsto (X',Y,Z)$ where explicitly

$$X' = \begin{cases} (F_1 X_1 + F_0 X_0 : -F_0 X_1) & \text{if this is in } \mathbb{P}^1, \\ (F_1 X_0 + F_2 X_1 : -F_2 X_0) & \text{otherwise} \end{cases}$$

and where we have suppressed Y and Z from the notation. We define σ_2 and σ_3 similarly.

If the curves $F_i(Y,Z) = 0$ don't intersect, then each σ_i is defined everywhere.

1.1 PICARD NUMBER 4

Without further restrictions, the Picard number of V will be 3 and V contains no (-2)-curves. Then the ample cone is $\mathcal{L}^+$. We now impose the condition that there is a point $(Q,Q') \in \mathbb{P}^1 \times \mathbb{P}^1$ such that

$$F_1(Q,Q') = F_2(Q,Q') = F_3(Q,Q') = 0$$

holds. Then $(X,Q,Q') \in V$ holds for all X, that is, V contains a line parallel to the X-axis, which is smooth and rational and therefore a (-2)-curve. If no further restrictions are imposed, then V will have Picard number 4.

REMARK 1.3. In a different paper, written by Baragar and Ronald [B–vL07], an example of such a surfaces is given where the authors prove that the Néron–Severi lattice has rank 4. From this example one can construct infinitely many.

We denote the divisor class of the line parallel to the X-axis by D_4. Since this line is a (-2)-curve and hence is the only effective divisor in its class, we abuse notation and write D_4 also for the line itself.

Remark 1.4. Since the classes D_2 and D_3 represent an elliptic fibration, their fibres over a point are degree 3 curves. In particular, their fibres above Q and Q' are each the union of the line D_4 and a conic.

Lemma 1.5. *The set* $\{D_1, D_2, D_3, D_4\}$ *forms a basis of* $\mathrm{NS}(V)$.

Proof. We already know that D_1, D_2 and D_3 are linearly independent. Since in the generic case the surface does not contain any (-2)-curves and therefore its real Néron–Severi vector space contains no negative self-intersection classes, the set $\{D_1, D_2, D_3, D_4\}$ is also linearly independent and therefore generates a sublattice of finite index m in $\mathrm{NS}(V)$. One checks the following intersection numbers:

$$D_i \cdot D_j = 2\delta_{ij}$$

for $i, j \in \{1, 2, 3\}$ and

$$D_4 \cdot D_i = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{if } i = 2, 3; \end{cases}$$

where $D_4 \cdot D_1 = 1$ is found since D_4 is a section of the elliptic fibration defined by D_1 and $D_4 \cdot D_{2,3} = 0$ is found by using the remark above and $-2 + 2 = 0$. The intersection matrix for the sublattice generated by $\{D_1, D_2, D_3, D_4\}$ has determinant -28, so $m = 1$ or $m = 2$ holds. If $m = 2$ holds, then $\mathrm{NS}(V)$ contains a class that only has coefficients 0 or $\frac{1}{2}$ with respect to $\{D_1, D_2, D_3, D_4\}$. Solving equations for the possibilities for such a class only leaves $D_2/2$ and $D_3/2$. Since both D_2 and D_3 are the class of an elliptic curve, which by definition are irreducible, we conclude $m = 1$. □

Remark 1.6. We've said that in the generic case, σ_i for $i = 1, 2, 3$ was defined everywhere. In this case σ_2 and σ_3 still are, but σ_1 is yet only defined on $V \setminus D_4$. It is possible to extend its definition to all of V and we refer to [Bar11, section 3] for the construction.

Proposition 1.7. *Denote with* T_i *the matrix representation of* σ_i^* *with respect to the basis* $\{D_1, \ldots, D_4\}$. *Then*

$$T_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, T_3 = \begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Proof. These are found from first calculating $T_i^t J = J T_i$ and then solving for T_i for $i = 1, 2, 3$. One calculates $\sigma_i^*(D_j) \cdot D_k$ using geometric means. For details we refer to [Bar11, Thm. 3.1]. □

The maps σ_i^* for $i = 1, 2, 3$ lie in $\mathcal{O}''$. Baragar also cites two different maps that do so, the map S that switches D_2 and D_3 and a map whose matrix representation is

$$T_4 := \begin{pmatrix} 1 & 8 & 8 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 4 & 4 & 1 \end{pmatrix}$$

and which supposedly was found by trial and error. From here on we abuse notation and also use T_i where we mean the map σ_i^* and likewise use T_4 where we mean the map with this matrix representation.

In fact, T_4 also comes from an automorphism of the surface, see Remark 2 of [Bar11].

PROPOSITION 1.8. *The maps S and T_4 are also in $\mathcal{O}''$.*

Proof. To show that a map contained in $\mathcal{O}^+$ (which S and T_4 are) is in fact contained in $\mathcal{O}''$ it suffices to find an ample divisor such that its image is again ample.

The divisor $D = D_1 + D_2 + D_3$ is ample and clearly $SD = D$ holds since S fixes D_1 and interchanges D_2 and D_3. We construct a different ample divisor that is fixed by T_4 to show $T_4 \in \mathcal{O}''$.

The divisors D_1 and D_4 are both the class of a curve, so both are irreducible. The divisor $D' := 2D_1 + D_4$ intersects non-negatively with each of its irreducible components, so also with each effective divisor. That means that D' is nef. We have $D_1 \cdot D_1 = 0$ so also D_1 is nef. In fact, D_1 lies on the boundary of the nef (and hence ample) cone. Since the nef cone is convex, the divisor $D'' := D' + D_1 = 3D_1 + D_4$ is nef and is therefore ample or lies on the boundary of the ample. Assume towards a contradiction that D'' lies on the boundary. Then by the lemma below there is a (-2)-curve N such that both $D_1 \cdot N = 0$ and $D' \cdot N = 0$ hold. An easy calculation shows that no such N may exist. Therefore D'' must be ample. Since T_4 fixes both D_1 and D_4, it fixes D''. □

LEMMA 1.9. *Let $\mathcal{C}$ be a closed convex cone. If $A \in \mathcal{C}$ holds and B and $A + B$ lie on $\partial\mathcal{C}$ then A, B (and consequently also $A + B$) lie on the same wall of $\mathcal{C}$.*

Proof. Left as an exercise. □

1.2 VISUALIZATIONS

Let R_{D_4} be the map that is reflection through D_4. We know that it is in $\mathcal{O}^+$ but not in $\mathcal{O}''$. We denote $G = \langle T_1, T_2, T_4, S, R_{D_4}\rangle$ and $G' = \langle T_1, T_2, T_4, S\rangle$. Remark that $T_3 = ST_2S$ holds, so also T_3 is contained in both G and G'.

We will see that $G = \mathcal{O}^+$ and $G' = \mathcal{O}''$ hold, but we first prove a weaker statement.

PROPOSITION 1.10. *G is of finite index in $\mathcal{O}^+$ and G' is of finite index in $\mathcal{O}''$.*

Proof. The second statement follows immediately from the first since there is the equality $G' = G \cap \mathcal{O}'' \subset \mathcal{O}$.

For the first statement, we construct a region $\mathcal{F}$ in $\mathcal{H}$ such that $\mathcal{F}$ is a fundamental domain for the action of G on $\mathcal{H}$. This region then contains a fundamental domain for the action of $\mathcal{O}^+$ on $\mathcal{H}$. If furthermore $\mathcal{F}$ has finite volume, then result follows.

To define this region $\mathcal{F}$, we use a pictorial representation of $\mathcal{H}$ and include the fixed point sets for (some of the) maps in G. For this, we remember that $\mathcal{H}$ is has a hyperbolic geometry. We first map $\mathcal{H}$ to a Poincaré ball which is then mapped to the Poincaré upper half space. To make a picture, we only consider the boundary of the last space (i.e. the $z = 0$ part). Lines in the picture represent planes in the upper half space and circles in the picture represent hemispheres in the upper half space. This construction is akin to the mapping from a hyperboloid to the upper half plane, where in the 2-dimensional case we need not take the last step for making the picture.

The resulting picture is Figure 2 from Baragars paper. We reference the lines and circles found there.

- The map T_1 is reflection through the hemisphere represented by the circle Γ_1.
- The map T_2 is a rotation of π around the line containing D_1 and D_3, i.e. it rotates everything from the one side of the plane represented by Γ_2 to the other side.
- Similarly for T_3 and the line Γ_3.
- Again similarly for T_4 and Γ_S.
- S reflects through the plane represented by Γ_S.
- R_{D_4} reflects through the hemisphere Γ'_1.
- $T_3 R_{D_4} T_3$ reflects through Γ'_2.
- $T_2 R_{D_4} T_2$ reflects through Γ'_3.
- Finally, $T_2 T_4 T_3 S = T_2 T_4 S T_2$ reflects through the plane Γ_5.

By construction, the region $\mathcal{F}$ which we define as the region bounded by the planes Γ_S, Γ_5 and Γ_2 and which lies above the hemispheres Γ'_1, Γ'_2 and Γ_1 is a fundamental domain for the action of G. □

PROPOSITION 1.11. *There are no* (-2)*-classes* C *such that the plane* $C^{\perp}$ *intersects the region* $\mathcal{F}$*.*

Remark that these C need not be (-2)-curves.

Proof. See [Bar11, Lem.5.1]. □

COROLLARY 1.12. *The region* $\mathcal{F}$ *is a subset of the nef cone.*

Proof. We first check that $D = D_1 + D_2 + D_3$ (which is ample) lies on the face of $\mathcal{F}$ given by the plane represented by Γ_S. Assume towards a contradiction that $\mathcal{F}$ contains a divisor D' that is neither ample, nor on either of the faces represented by Γ'_1 or Γ'_2. Since $\mathcal{F}$ is convex, the line segment between D and D' lies entirely in $\mathcal{F}$. By assumption, this line segment must cross the boundary of the ample cone, so it must intersect a plane defined by $N^{\perp}$ for some (-2)-curve N. By the lemma above, no such N may exist. □

PROPOSITION 1.13. $G = \mathcal{O}^+$ *and* $G' = \mathcal{O}''$*.*

Proof. See [Bar11, Lem.5.3]. In the first line of the proof, I believe it should say $T' \in G$. There are still some more things that I don't understand about the proof. I have to think about them some more or perhaps ask the author. □

We have found the group $\mathcal{O}''$ which is the group of effective isometries of $\mathrm{NS}(V)$. However, we do not know what isometries the transcendental lattice has and therefore cannot yet find the automorphism group of V. We only know that $\mathcal{O}''$ is of finite index in it.

References

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