An algorithm to compute automorphism groups of K3 surfaces

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Introduction

Let X be a K3 surface, and let denote by S_X its Néron-Severi group. Let $\mathcal{O}(S_X)$ be the orthogonal group of S_X, that is, the group of isometries of S_X. In these notes we present algorithm 6.1 in [Shi14]. This algorithm computes, under some assumptions on S_X, a set of generators of the image of the natural morphism

$$\varphi_X : \mathrm{Aut}(X) \to \mathcal{O}(S_X)$$

from the group of automorphisms of X to the gorup of isometries of S_X.

This is not the complete output. In fact the algorithm provides also other information about the surface X. For a more detailed discussion of the output see Section 4.

In these notes we skip most of the theory that is instead provided in the paper.

1 Notation

Let S be an even hyperbolic lattice with a fixed positive cone $\mathcal{P}_S$. Let $\mathcal{O}(S)$ the set of isometries of S and define $\mathcal{O}^+(S)$ as the subgroup of $\mathcal{O}(S)$ given by all the isometries preserving $\mathcal{P}_S$.

Let $G \subseteq \mathcal{O}^+(S)$ be a finite index subgroup of $\mathcal{O}^+(S)$, and assume that G satisfies the following condition:

[G] There exists an algorithm by which we can determine, for a given $g \in \mathcal{O}^+(S)$, whether $g \in G$ or not.

Let $\mathbb{L}$ be an even hyperbolic lattice of rank $n \in \{10, 18, 26\}$.

Remark 1.1. For example, we can consider $\mathbb{L} \cong U \oplus E_8(-1)^{\oplus i}$ with $i \in \{1,2,3\}$ and

$$U = \left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}\right).$$

Such $\mathbb{L}$ is unique up to isomorphisms.

We assume that

[SG1] S is primetively embedded into $\mathbb{L}$.

Let $R = S^{\perp} \subset \mathbb{L}$ the orthogonal complement of S inside $\mathbb{L}$. Then R is a negative definite lattice. We require that

[SG2] If $n = 26$ then R cannot be embedded into the Leech lattice Λ_{24}.

We also ask that

[SG3] all the elements of G are liftable to elements of $\mathcal{O}(\mathbb{L})$.

We now introduce the notion of chambers and walls of S. The same notion can be generalised to any lattice, and in particular also to $\mathbb{L}$

Definition 1.2. *Let $\mathcal{R}_S = \{r \in S \mid r^2 = -2\}$ be the set of -2 vectors in S, it induces the set of walls $\mathcal{R}_S^* = \{(r^{\perp}) \subset S \mid r \in \mathcal{R}_S\}$. A $\mathcal{R}_S^*$ chamber of S is a chamber of S whose walls are in $\mathcal{R}_S^*$.*

Since $R \oplus S$ is of finite index inside $\mathbb{L}$, we have that $\mathbb{L}_{\mathbb{Q}}$ embeds into $R_{\mathbb{Q}} \oplus S_{\mathbb{Q}}$, and therefore we can consider the projection

$$\mathrm{pr}_S: \mathbb{L}_{\mathbb{Q}} \to S_{\mathbb{Q}}$$

sending a vector x to its S-component x_S. This embedding allows us to give the following definition.

Definition 1.3. *We define the set of -2 vectors of $\mathbb{L}$ that are S-negative as*

$$\mathcal{R}_{\mathbb{L}|S} := \{r_S \in S_{\mathbb{Q}} \mid r \in \mathcal{R}_{\mathbb{L}} \text{ and } r_S^2 < 0\}.$$

Clearly, $\mathcal{R}_S \subseteq \mathcal{R}_{\mathbb{L}|S}$. Therefore, if N is a $\mathcal{R}_S^*$-chamber, then it is a union of $\mathcal{R}_{\mathbb{L}|S}^*$-chambers. We define the notion of N-chain and level of a $\mathcal{R}_{\mathbb{L}|S}^*$-chamber inside N.

Definition 1.4. *Let N be a $\mathcal{R}_S^*$-chamber and fix a $\mathcal{R}_{\mathbb{L}|S}^*$-chamber D_0 inside N. Let D be another $\mathcal{R}_{\mathbb{L}|S}^*$-chamber inside N. An N-chain is a finite sequence $D^{(0)}, \ldots, D^{(l)}$ of $\mathcal{R}_{\mathbb{L}|S}^*$-chambers inside N such that $D^{(i)}$ is adjacent to $D^{(i+1)}$. Since N is connected, there is a N-chain $D^{(0)}, \ldots, D^{(l)}$ such that $D^{(0)} = D_0$ and $D^{(l)} = D$. The level of D is the minimum of the length of such chains.*

We introduce different notions of defining set of a chamber.

Definition 1.5. *Let* Δ *be a subset of* $\mathcal{N}_{\mathbb{L}} = \{r \in \mathbb{L}_{\mathbb{Q}} \mid r^2 < 0\}$*, then we define the* Δ*-positive cone*

$$\Sigma_{\mathbb{L}}(\Delta) := \{x \in \mathbb{L}_{\mathbb{R}} \mid \forall r \in \Delta,\ \langle x, r\rangle_{\mathbb{L}} \geq 0\}.$$

Let $\mathcal{R}_{\mathbb{L}} = \{r \in \mathbb{L} \mid r^2 = -2\}$ *and consider an* $\mathcal{R}_{\mathbb{L}}^*$*-chamber* D*. The subset* $\Delta \subset \mathcal{N}_{\mathbb{L}} \cap \mathbb{L}^\vee$ *is a defining set of* D *if*

$$D = \Sigma_{\mathbb{L}}(\Delta) \cap \mathcal{P}_{\mathbb{L}}.$$

It is minimal if

- $\forall r \in \Delta$, $(r)^\perp$ *is a wall of* D*,*
- *if* r, r' *are distinct elements of* Δ*, then* $(r)^\perp \neq (r')^\perp$*.*

If D *is a* $\mathcal{R}_{\mathbb{L}}^*$*-chamber we define its* $\mathcal{R}_{\mathbb{L}}^*$*-minimal defining set* $\Delta_{\mathcal{R}_{\mathbb{L}}}(D)$ *to be a minimal defining set such that* $\Delta_{\mathcal{R}_{\mathbb{L}}}(D) \subset \mathcal{R}_{\mathbb{L}}$ *and if* $v \in \Delta_{\mathcal{R}_{\mathbb{L}}}(D)$*, then* $\alpha v \notin \mathcal{R}_{\mathbb{L}}$ *for any* $\alpha \in]0,1[$*.*
We define its primitively minimal defining set $\Delta_{\mathbb{L}^\vee}(D)$ *to be the minimal defining set of* D *such that every element in it is primitive in* $\mathbb{L}^\vee$*.*

Remark 1.6. Let D be an $\mathcal{R}_{\mathbb{L}}^*$-chamber. A careful reader might have notice that its $\mathcal{R}_{\mathbb{L}}^*$-minimal defining set $\Delta_{\mathcal{R}_{\mathbb{L}}}(D)$ and its primitively minimal defining set $\Delta_{\mathbb{L}^\vee}(D)$ are in fact equal. This is due to the two following facts: if v is in $\mathcal{R}_{\mathbb{L}}$ then no multiples of v, except -1, lie in $\mathcal{R}_{\mathbb{L}}$; the lattice $\mathbb{L}$ is unimodular and therefore $\mathbb{L}^\vee = \mathbb{L}$. In [Shi14] the notion of $\mathcal{V}$-minimal and primitively minimal defining sets are given in a more general context: the lattice $\mathbb{L}$ is any hyperbolic even lattice (not necessarely unimodular) and $\mathcal{V}$ is any subset of $\mathcal{N}_{\mathbb{L}} \cup \mathbb{L}^\vee$ (not necessarely $\mathcal{R}_{\mathbb{L}}$).

Nevertheless, we keep this notation, for an easier combined use of these notes and the original paper.

A useful tool to deal with chambers is given by the Weyl vectors.

Definition 1.7. *Let* D *be a* $\mathcal{R}_{\mathbb{L}}^*$*-chamber, we say that* $w \in \mathbb{L}$ *is a Weyl vector of* D *if the* $\mathcal{R}_{\mathbb{L}}$*-minimal defining set of* D *is*

$$\Delta_{\mathcal{R}_{\mathbb{L}}}(D) = \{r \in \mathcal{R}_{\mathbb{L}} \mid \langle w, r\rangle = 1\}.$$

Let now D *be an* $\mathcal{R}_{\mathbb{L}|S}^*$*-chamber and* D' *a* $\mathcal{R}_{\mathbb{L}}^*$*-chamber such that* $D = D' \cap \mathcal{P}_S$*. By an abuse of notation, we say that a Weyl vector* $w \in \mathbb{L}$ *of* D' *is also a Weyl vector of* D*.*

It is a fact that every $\mathcal{R}^*_{\mathbb{L}}$-chamber admits a Weyl vector (see [Shi14, Theorem 4.2]).

Notice that a Weyl vector completely determines a chamber. We will use the Weyl vectors to store the list of a complete set of representatives of G-congruence classes of $\mathcal{R}^*_{\mathbb{L}|S}$-chambers contained in N.

Let D be a $\mathcal{R}^*_{\mathbb{L}}$-chamber and $w \in \mathbb{L}$ a Weyl vector of D. Then we have that

$$D = \Sigma_{\mathbb{L}}(\Delta_{\mathcal{R}_{\mathbb{L}}}(D)) \cap \mathcal{P}_{\mathbb{L}} \ .$$

We define

$$\Delta_w = \{r \in \Delta_{\mathcal{R}_{\mathbb{L}}}(D) \mid r_S^2 < 0\} = \{r \in \Delta_{\mathcal{R}_{\mathbb{L}}}(D) \mid (r)^\perp \cap \mathcal{P}_S \neq 0\}.$$

It follows that we can write its orthogonal projection on S as

$$\mathrm{pr}_S(\Delta_w) = \{r_S \mid r \in \Delta_w\}$$

and we have that

$$D \cap \mathcal{P}_S = \Sigma_S(\mathrm{pr}_S(\Delta_w)) \cap \mathcal{P}_S.$$

Therefore, if S is non-degenerate with respect to D, i.e., $(D \cap \mathcal{P}_S)^\circ \neq \varnothing$, then $\mathrm{pr}_S(\Delta_w)$ is a defining set of the $\mathcal{R}^*_{\mathbb{L}|S}$-chamber $D' = D \cap \mathcal{P}_S$.

Definition 1.8. *Let G be a finite index subgroup of $\mathcal{O}^+(S)$ and let D be a $\mathcal{R}^*_S$-chamber, then we put*

$$\mathrm{Aut}_G(D) := \{g \in G \mid D^g = D\}.$$

2 The input

In this section we explain what is the input we would in practice give to the algorithm.

Let X be a complex K3 surface. Let $S = S_X$ denote its Néron-Severi lattice. Let $\mathcal{P}_S$ be the connected component of the positive cone of S containing an ample class, and let $N = N_X$ be the $\mathcal{R}^*_S$-chamber given by

$$N := \mathcal{P}_S \cap \mathrm{Nef}(X),$$

where $\mathrm{Nef}(X)$ is the Nef cone of X, that is, the cone of $S_X \otimes \mathbb{R}$ given by

$$\mathrm{Nef}(X) := \{x \in S_X \otimes \mathbb{R} \mid \langle x, C\rangle \geq 0 \text{ for any curve } C \subset X\}.$$

Remark 2.1. Notice that $N = \overline{\mathrm{Amp}(X)}$ (see [Laz04, Theorem 1.4.23]).

Since X is projective, its Néron-Severi lattice S is a hyperbolic sublattice of $H^2(X,\mathbb{Z}) \cong \Lambda_{K3}$. Let $T = T_X$ denote the orthogonal complement of S inside $\Lambda_K 3$, that is, *the transcendental lattice* of X. Let $\mathcal{O}_H(T)$ denote the group of Hodge isometries of T, that is, the set

$$\mathcal{O}_H(T) = C_X := \{g \in \mathcal{O}(T) \mid H^{2,0}(X)^g = H^{2,0}(X)\}.$$

Let $G = G_X$ be the subgroup of $\mathcal{O}^+(S)$ given by

$$G_X := \{g \in \mathcal{O}^+(S) \mid \delta_{ST}(\eta_S(g)) \in \eta_T(\mathcal{O}_H(T))\}$$

(cf. Figure 1).

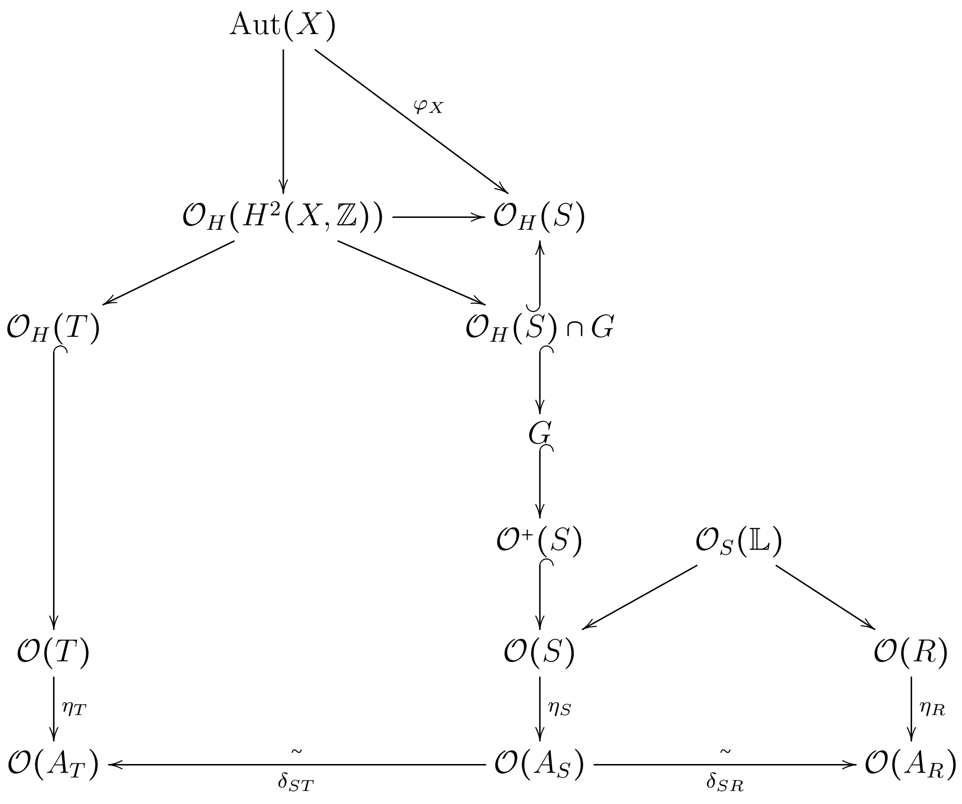

Figure 1: Notice that $\mathcal{O}_H(S) \cap G = Im\ (\varphi_X)$.

In order to succesfully apply Algorithm 3.1, we need the following conditions to hold:

1. The subgroup G satisfies the condition [G].

2. The lattice S satisfies conditions [SG1], [SG2] and [SG3].

3. We can find a Weyl vector of an $\mathcal{R}^*_{\mathbb{L}|S}$-chamber D_0 contained in the $\mathcal{R}^*_S$-chamber N.

We are going to see whether these conditions are satisfied.

2.2 Requirement (1)

By the definition of G, we have that condition [G] is satisfied if we have an explicit description of the canonical isomorphism $A_S \to A_T$ and the morphisms $\eta_S: \mathcal{O}(S) \to \mathcal{O}(A_S)$ and $\eta_R: \mathcal{O}(R) \to \mathcal{O}(A_R)$.

If $\mathcal{O}_H(T) = \{\pm 1\}$ then the condition for g to be an element of G reduces to the condition $\eta_S(g) = \pm 1$. In order to verify this condition it is enough to have an explicit description of η_S

2.3 Requirement (2)

By [Shi14, Propostion 8.1] we have that condition [SG1] always holds for $S = S_X$.

Remark 2.4. Even though such a primitive embedding always exists, in practice it might be hard to compute it explicitely.

If $n = 10, 18$, then condition [SG2] is void. If $n = 26$ and an explicit embedding $S \hookrightarrow \mathbb{L}$ is given, then it is possible to check condition [SG2]. In practice, to ensure that condition [SG2] holds, it is enough to check that $\mathcal{R}_{R(-1)} \neq \emptyset$.

By the definition of G, we have that

$$\eta_S(G) = \eta_S(\mathcal{O}^+(S)) \cap \delta_{ST}^{-1}(\eta_T(\mathcal{O}_H(T)).$$

The condition [SG3] is equivalent to the condition

$$\eta_S(G) \subseteq \delta_{SR}^{-1}(\eta_R(\mathcal{O}_R)).$$

It follows that if $\delta^{-1}(\eta_T(\mathcal{O}_H(T)) \subseteq \delta_{SR}^{-1}(\eta_R(\mathcal{O}_R))$ then [SG3] is satisfied. This condition is fulfilled in two cases:

1. the map η_R is surjective, or

2. $\mathcal{O}_H(T) = \{\pm 1\}$.

2.5 Requirement (3)

The third requirement is the most technical one. In [Shi14, Section 8.3] two methods are shown to obtain a Weyl vector of an $\mathcal{R}^*_{\mathbb{L}|S}$-chamber D_0 contained in the $\mathcal{R}^*_S$-chamber N. Both methods make use of an ample class of S_X. The details can be found in [Shi14, Section 8.3] and [Shi14, Section 4].

3 The algorithm

Let X be a K3 surface and let $\mathbb{L}$, S, G, N, D_0 and w_0 be defined as in Section 2.

The Algorithm 3.1 computes the following objects:

- a finite set Γ of generators of
$$Im\ \varphi_X = \mathrm{Aut}_G(N) := \{g \in G \mid N^g = N\};$$
- a finite set $\mathbb{D}$ of $\mathcal{R}^*_{\mathbb{L}|S}$-chambers contained in N such that, if F_N denotes the union $\bigcup_{D_i \in \mathbb{D}} D_i$, then for each element $v \in N$ there is an element $g \in \mathrm{Aut}_G(N)$ such that $v^g \in F_N$. In fact, $\mathbb{D}$ will be a complete set of representatives of G-congruence classes of $\mathcal{R}^*_{\mathbb{L}|S}$-chambers and it is a union of nonempty subsets $\mathbb{D}_l$, whose elements are $\mathcal{R}^*_{\mathbb{L}|S}$-chambers of level l;
- a subset $\mathcal{B} \subset \mathcal{R}_S$ such that, if r is an element of the $\mathcal{R}_S$-minimal defining set $\Delta_{\mathcal{R}_S}(N)$ of N, then there exists an element $g \in \mathrm{Aut}_G(N)$ such that $r^g \in \mathcal{B}$.

Algorithm 3.1 (Algorithm 6.1)**.** *Let* $\mathbb{L}$, S, G, N, D_0 *and* w_0 *be defined as in Section 2. The algorithm computes a finite set of generators* Γ *for* $Im\ \varphi_X$*, a complete set of representatives* $\mathbb{D}$ *of* G*-congruence classes of* $\mathcal{R}^*_{\mathbb{L}|S}$*-chambers inside* N*, and a fundamental domain* F_N *for the action of* $\mathrm{Aut}(X)$ *on* $\mathrm{Nef}(X)$*.*

1. *Set* $\Gamma := \{\}$, $\mathbb{D}_0 := \{D_0\}$, $\mathcal{B} := \{\}$*;*
2. *compute* $\Delta_{S^\vee}(D_0)$ *from* w_0 *using Algorithm 3.2;*
3. *perform Algorithm 3.3 using* $l = 0$*.*

In what follows we describe the auxiliary algorithms needed to perform Algorithm 3.1.

Algorithm 3.2 (5.11)**.** *Let D_0 be a $\mathcal{R}^*_{\mathbb{L}|S}$-chamber and w_0 be a Weyl vector of D_0. This algorithm computes $\Delta_{S^\vee}(D)$.*

1. *Compute Δ_{w_0} using Algorithm 3.4;*

2. *compute $\mathrm{pr}_S(\Delta_{w_0})$;*

3. *compute $\Delta_{S^\vee}(D)$ from $\mathrm{pr}_S(\Delta_{w_0})$ using Algorithm 3.5.*

Algorithm 3.3 (Procedure `adj()`)**.** *Let l be a non-negative integer, N a $\mathcal{R}^*_S$-chamber and D_0 a $\mathcal{R}^*_{\mathbb{L}|S}$. Assume that*

- *for each $\lambda = 0, ..., l$ a non-empty finite set $\mathbb{D}_\lambda$ of $\mathcal{R}_{\mathbb{L}|S}$-chambers is given,*

- *each $D \in \mathbb{D}_\lambda$ is contained in N and it is of level λ,*

- *if D, D' are distinct elements in $\bigcup_{\lambda=0}^{l} \mathbb{D}_\lambda$, then they are not conjugated under the action of G,*

- *for any D in $\bigcup_{\lambda=0}^{l} \mathbb{D}_\lambda$ the set $\Delta_{S^\vee}(D)$ has been computed.*

The algorithm compute $\mathbb{D}_{l+1}$ and $\Delta_{S^\vee}(D)$ for each $D \in \mathbb{D}_{l+1}$.

In what follows we will denote by $D_{k+1}, ..., D_{k+m}$ the elements of $\mathbb{D}_l$, where $k = \sum_{\lambda=0}^{l-1} \#\mathbb{D}_\lambda$ and $m = \#\mathbb{D}_l$. If $l = 0$ we set $k = -1$.

1. *Set $\mathbb{D}' := \{\}$ and $j := k + m + 1$;*

2. *for each element D_i in $\mathbb{D}_l$ do:*

   (a) *compute $\mathrm{Aut}_G(D_i)$ from $\Delta_{S^\vee}(D_0)$ by using Algorithm 3.11 and append a finite set of generators of $\mathrm{Aut}_G(D_i)$ to Γ.*

   (b) *Notice that $\mathrm{Aut}_G(D_i)$ acts on $\Delta_{S^\vee}(D_0)$. Therefore decompose $\Delta_{S^\vee}(D_i)$ int $\mathrm{Aut}_G(D_i)$-orbits $o_1, ..., o_t$. Let ν be an element in $\{1, ..., t\}$. Since $\mathcal{R}^*_S$ is G-invariant, the set*

   $$o^*_\nu := \{(v)^\perp \mid v \in o_\nu\}$$

   is either disjoint from $\mathcal{R}^*_S$ or entirely contained in it. Let v be an element of o_ν. Since v is primitive in $S^\vee$, we have that o^*_ν is contained in $\mathcal{R}^*_S$ if and only if there is a positive integer α such that $\alpha^2 v^2 = -2$ and $\alpha v \in S$;

   (c) *for each orbit o_ν such that o^*_ν is disjoint from $\mathcal{R}^*_S$ do:*

      i. *choose a vector $v \in o_\nu$ and compute a Weyl vector $w' \in \mathbb{L}$ of the $\mathcal{R}^*_{\mathbb{L}|S}$-chamber D' adjacent to D_i along the wall $(v)^\perp$ using Algorithm 3.8;*

ii. *compute* $\Delta_{S^\vee}(D')$ *using Algorithm 3.2. Since* $(v)^\perp$ *is not in* $\mathcal{R}^*_S$ *and* $D_i \subset N$*, we have that* $D' \subset N$*. Since* D_i *is of level* l*, we see that* D' *is of level* $\leq l+1$*;*

iii. *Using Algorithm 3.10, determine whether* D' *is* G*-congruent to an* $\mathcal{R}^*_{\mathbb{L}|S}$*-chamber* D'' *in*

$$\tilde{D} := \mathbb{D}_0 \cup \ldots \cup \mathbb{D}_l \cup \mathbb{D}' = \{D_0, \ldots, D_{j-1}\}$$

or not.

iv. *If* $D' = D''^h$ *for some* $D'' \in \tilde{\mathbb{D}}$ *and some* $h \in G$ *then append* h *to* Γ*.*

v. *If there exist no such* D'' *and* h *then put* $D_j := D'$*, append* D_j *to* $\mathbb{D}'$ *and increment* j *by* 1*. Notice that in this case we we have that the level of* D' *is exactly* $l+1$*.*

(d) *For each orbit* o_ν *such that* $o^*_\nu \subset \mathcal{R}^*_S$*, choose a vector* $v \in o_\nu$*, find a positive integer* α *such that* $r := \alpha v \in \mathcal{R}_S$ *and append* r *to* $\mathcal{B}$*.*

3. *If* $\mathbb{D}' \neq \varnothing$*, then put* $\mathbb{D}_{l+1} := \mathbb{D}'$ *and execute the algorithm for* $l+1$*.*

4. *If* $\mathbb{D}' = \varnothing$*, then set*

$$\mathbb{D} := \mathbb{D}_0 \cup \ldots \cup \mathbb{D}_l$$

and terminate.

Algorithm 3.4 (5.8)**.** *Let* D *be a* $\mathcal{R}^*_{\mathbb{L}}$*-chamber and* $w \in \mathbb{L}$ *a Weyl vector of* D*. This algorithm computes the set* Δ_w*.*

1. *compute* $w_S \in S^\vee$*,* $w_R \in R^\vee$ *and the set*

$$n_R := \{c \in \mathbb{Q} \mid d_R c \in \mathbb{Z}\ ,\ d_R^2 c \in 2\mathbb{Z}\ ,\ -2 < c \leq 0\ \}$$

where $d_R = \#A_R$*. Set* $\Delta' := \{\}$*.*

2. *for each* $c \in n_R$ *compute*

$$R^\vee[c] := \{v \in R^\vee \mid v^2 = c\}$$

using Algorithm 3.6 and compute also

$$a_R[c] := \{\langle w_R, v\rangle_R \mid v \in R^\vee[c]\}.$$

3. *For each* $n' \in n_R$ *and* $a' \in a_R[n']$*, compute*

$$S^\vee[n', a'] := \{v \in S^\vee \mid \langle w_S, v\rangle_S = 1 - a'\ ,\ v^2 = -2 - n'\}$$

using Algorithm 3.7.

4. *For each* $n' \in n_R$, $v_R \in R^\vee[n']$ *and* $v_S \in S^\vee[n', a']$, *where* $a' = \langle w_R, v_R \rangle_R$, *we check whether* $v_S + v_R \in S^\vee \oplus R^\vee$ *is in* $\mathbb{L}$. *If it is so, we append* $v_S + v_R$ *to* Δ'.

5. *Output* $\Delta_w := \Delta'$.

Algorithm 3.5 (3.17)**.** *Let* Δ *be a finite defining set of a* $\mathcal{R}_S^*$*-chamber* D. *This algorithm computes* $\Delta_{S^\vee}(D)$.

1. *Set* $\Delta_1 = \{\}$ *and* $\Delta_2 = \{\}$.

2. *For each* $v \in \Delta$ *compute the integer*

$$a_v := \max\{a \in \mathbb{Z}_{>0} \mid v/a \in S^\vee\}.$$

   Append v/a_v *to* Δ_1. *(Then* $D = \Sigma_S(\Delta_1) \cap \mathcal{P}_S$ *and if* $v, v' \in \Delta_1$, *then* $v = v'$ *if and only if* $(v)^\perp = (v')^\perp$*)*

3. *For each* $v \in \Delta_1$ *do*

   (a) *if* $\Delta_1 \setminus \{v\}$ *does not span* $\mathbb{L} \otimes \mathbb{R}$, *then appen* v *to* Δ_2*;*

   (b) *else, solve the linear programming problem*

$$\begin{cases} \text{minimize } \langle v, x \rangle, \\ \text{subject to } \langle v', x \rangle \geq 0 \text{ for any } v' \in \Delta_1 \setminus \{v\} \end{cases}$$

   The solution is either 0 *or* $-\infty$.

   (c) *If the solution is* $-\infty$ *then append* v *to* Δ_2.

4. *Output* $\Delta_{S^\vee}(D) := \Delta_2$.

Algorithm 3.6 (2.1)**.** *Let* Q *be s positive definite matrix of size* n *with raional entries,* l *a column vector of lenght* n *with rational entries and* c *a rational number. There is an algorithm to compute the list of* $x \in \mathbb{Z}^n$ *such that*

$$xQ^t x + 2x \cdot l + c = 0.$$

Algorithm 3.7 (2.2)**.** *Let* L *be a hyperbolic lattice,* $v \in L_{\mathbb{Q}}$ *such that* $v^2 > 0$. *Let* $\alpha \in \mathbb{Q}$, $d \in \mathbb{Z}$, *then we can compute*

$$\{x \in L \mid \langle x, v \rangle = \alpha\,,\ \langle x, x \rangle = d\}.$$

Algorithm 3.8 (5.14)**.** *Let* D *be a* $\mathcal{R}_{\mathbb{L}|S}^*$*-chamber and* $w \in \mathbb{L}$ *a Weyl vector of* D. *Let* v *be a given element of* $\Delta_{S^\vee}(D)$. *This algorithm computes a Weyl vector* w' *of the* $\mathcal{R}_{\mathbb{L}|S}^*$*-chamber* D' *adjacent to* D *along the wall* $(v)^\perp$.

1. *Compute the set*
$$P_v := \{r \in \mathcal{R}_{\mathbb{L}} \mid r_S \in \mathbb{R}v\}$$
using Algorithm 3.9.

2. *Choose a complete set of representatives* $P'_v = \{r_1, ..., r_N\}$ *of* $P_v/\{\pm 1\}$.

3. *Choose a vector* $u \in \mathbb{L}_{\mathbb{Q}}$ *such that if* $i \neq j$ *then*
$$\frac{\langle u, r_i\rangle_{\mathbb{L}}}{\langle w, r_i\rangle_{\mathbb{L}}} \neq \frac{\langle u, r_j\rangle_{\mathbb{L}}}{\langle w, r_j\rangle_{\mathbb{L}}}$$

4. *Sort the elements* r_i *of* P'_v *so that if* $i < j$ *then*
$$\frac{\langle u, r_i\rangle_{\mathbb{L}}}{\langle w, r_i\rangle_{\mathbb{L}}} < \frac{\langle u, r_j\rangle_{\mathbb{L}}}{\langle w, r_j\rangle_{\mathbb{L}}}.$$

5. *Return* $w^{s_1 \cdot \ldots \cdot s_N}$, *where* $s_i \in \mathcal{O}^+(\mathbb{L})$ *is the reflection with respect to* r_i.

Algorithm 3.9 (5.13)**.** *Assume* $v \in \mathcal{N}_S \cap S^\vee$ *is given. This algorithm computes the set*
$$P_v := \{r \in \mathcal{R}_{\mathbb{L}} \mid (v)^\perp \subseteq (r)^\perp\} = \{r \in \mathcal{R}_{\mathbb{L}} \mid r_S \in \mathbb{R}v\}.$$

1. *Set* $P := \{\}$.

2. *Compute the set* $\alpha_v := \{\alpha \in \mathbb{Q} \mid \alpha v \in S^\vee$ *and* $\alpha^2 v^2 \geq -2\}$.

3. *For each* $\alpha \in \alpha_v$ *let* $c = -2 - \alpha^2 v^2$ *and consider*
$$R^\vee[c] = \{u \in R \mid v^2 = c\}.$$

4. *For each* $\alpha \in \alpha_v$ *and* $u \in R^*[c]$ *determine whether* $r := \alpha v + u \in S^\vee \oplus R^\vee$ *is an element of* $\mathbb{L}$ *or not.*

5. *If it is, then append* $r \in \mathbb{L}$ *to* P.

6. *Output* $P_v := P$.

Algorithm 3.10 (3.19)**.** *Let* D *and* D' *be two* $\mathcal{R}^*_{\mathbb{L}|S}$*-chambers, and suppose* $\Delta_{S^\vee}(D)$ *and* $\Delta_{S^\vee}(D')$ *are given. Let* l *denote the rank of* S. *This algorithm determine whether* D *and* D' *are* G*-congruent.*

1. *Fix an element* $[v_1, ..., v_l] \in \Delta^l_{S^\vee}$ *that forms a basis of* $S_{\mathbb{Q}}$.

2. *For each l-tuple $[v'_1, ..., v'_l]$ of distinct elements of $\Delta_{S^\vee}$ compute the linear transformation g of $S_\mathbb{Q}$ such that $v_i^g = v'_i$ for any $i = 1, ..., l$.*

3. *If $g \in G$ and it induces a bijection from $\Delta_{S^\vee}(D)$ to $\Delta_{S^\vee}(D')$, then D and D' are G-congruent.*

4. *Else, D and D' are not G-congruent.*

Algorithm 3.11 (3.18)**.** *Let D be a $\mathcal{R}^*_{\mathbb{L}|S}$-chamber and suppose $\Delta_{S^*}(D)$ is given. This algorithm computes all the elements of $\mathrm{Aut}_G(D)$. Let l denote the rank of S.*

1. *Fix an element $[v_1, ..., v_l] \in \Delta^l_{S^\vee}$ that forms a basis of $S_\mathbb{Q}$.*

2. *Set $A = \{\}$.*

3. *For each l-tuple $[v'_1, ..., v'_l]$ of distinct elements of $\Delta_{S^\vee}$ compute the linear transformation g of $S_\mathbb{Q}$ such that $v_i^g = v'_i$ for any $i = 1, ..., l$.*

4. *If $g \in G$ and it induces a permutation of $\Delta_{S^\vee}(D)$ then append g to A.*

5. *Return $\mathrm{Aut}_G(D) := A$.*

4 The output

Proposition 4.1. *The Algorithm 3.1 performed with $\mathbb{L}$, S, G, N, D_0 and w_0 defined as in Section 2 terminates.*

Proof. See [Shi14, Proposition 6.2]. The proof relies on the fact that there are only finitely many G-congruence classes of $\mathcal{R}^*_{\mathbb{L}|S}$-chambers inside N. □

Reading the description of the Algorithm 3.1, one can see that its output consists of a set Γ of elements of $\mathrm{Aut}_G(N)$, a collection $\mathbb{D}$ of $\mathcal{R}^*_{\mathbb{L}|S}$-chambers inside N, and a set $\mathcal{B}$ of elements of $\mathcal{R}_S$

Proposition 4.2. *Let $\mathbb{L}$, S, G, N, D_0 and w_0 be defined as in Section 2. Let Γ, $\mathbb{D}$, and $\mathcal{B}$ be the output of 3.1 performed taking $\mathbb{L}$, S, G, N, D_0, w_0. Then the following statements hold:*

1. *The group $\mathrm{Aut}_G(N)$ is generated by Γ.*

2. *For any $v \in N$ there exists an element $g \in \mathrm{Aut}_G(N)$ such that $v^g \in F_N$.*

3. *For any $r \in \Delta_{\mathcal{R}_S(N)}$ there exists an element $g \in \mathrm{Aut}_G(N)$ such that $r^g \in \mathcal{B}$.*

.

Proof. See [Shi14, Proposition 6.3]. □

Remark 4.3. If $\mathrm{Aut}_G(D) = \{1\}$ for any $D \in \mathbb{D}$, then F_N is a fundamental domain for the action of $\mathrm{Aut}_G(N)$ on N.

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