Elliptic surfaces – Erik Visse

These are the notes from the seminar on K3 surfaces and their automorphisms held in Leiden at the end of 2014 and start of 2015. The website for this seminar can be found at `http://pub.math.leidenuniv.nl/~vissehd/k3seminar/`.

1 Introduction

Today's lecture will not be about K3 surfaces (but we will mention them every now and then), but on surfaces admitting an elliptic fibration, so-called elliptic surfaces.

The notes for these lectures where based on parts of the first few chapters of [SchSh10] and on parts of [BHPV04] chapter V.

Definition 1.1. Let $f : S \to C$ be a morphism between a surface S and a curve C both smooth and projective over some field k. Then we say (S, f), or in short S is an *elliptic surface* if over the algebraic closure $\overline{k}$ one has the following:

(1) for almost all closed points $x \in C$, the fibre S_x is a smooth curve of genus 1,

(2) no fibres contain (-1)-curves.

Remark 1.2. Such surfaces are not necessarily minimal, i.e. they may still contain (-1)-curves.

Example 1.3. Consider two homogeneous cubics g and h in $\mathbb{P}^2$ that have no common factor such that each of the three variables of $\mathbb{P}^2$ occurs in at least one of g and h. Consider the pencil $S = \{\lambda g + \mu h = 0\} \subset \mathbb{P}^2 \times \mathbb{P}^1$ and the map to $\mathbb{P}^1$ that is given by projection on the second factor. Then S is an elliptic surface and it is isomorphic to the blow-up of $\mathbb{P}^2$ in the nine base points of $\lambda g + \mu h = 0$, so S does contain (-1)-curves; they are not fibre components but sections.

Definition 1.4. A *section* of an elliptic surface $f : S \to C$ is a morphism $\sigma : C \to S$ such that $f \circ \sigma = \mathrm{id}_C$ holds.

From now on, we further assume our elliptic surfaces to satisfy the following two conditions, again over the algebraic closure $\overline{k}$:

(3) each elliptic surface has a section,

(4) each elliptic surface has a non-smooth fibre.

The last condition in particular implies that S is not a product of an elliptic curve and the base curve C.

We regularly take a sum over all fibres (above closed points of C) and without further remark understand this to actually be a finite sum where only the singular fibres contribute non-trivial terms. This should also always be clear from context.

Definition 1.5. For each elliptic surface, we choose a favoured section σ_0 and write $\overline{O}$ for its image in S. We call σ_0 and $\overline{O}$ the *zero section*, where the context should make clear which of the two we mean.

Fact 1.6. The generic fibre $E = S_\eta$, the fibre above the generic point η of C is an elliptic curve over the function field $k(C)$, where its zero is the unique intersection point of E and $\overline{O}$.

From now on, we write $K = k(C)$.

There exist the following correspondence between sections and K-rational points on E:

Proposition 1.7. *There is a bijection* $\{$*sections of* $f : S \to C\} \leftrightarrow E(K)$*.*

Proof. Given a section σ, let $P_\sigma = E \cap \sigma(C)$ be the unique intersection point of E and $\sigma(C)$. It turns out that P_σ is defined over K.

Given a $E(K)$-point P, let $\Gamma = \overline{P}$ be the Zariski closure of P in S. Then $f|_\Gamma : \Gamma \to C$ is a birational morphism. By Zariski's main theorem, this morphism is in fact an isomorphism, so letting $i : \Gamma \to S$ be the embedding of Γ in S, the morphism $i \circ (f|_\Gamma)^{-1} : C \to S$ is a section. □

We consider the following lemma, which will be useful throughout the talk.

Lemma 1.8 (Zariski, [BHPV04). *III.8.2] Let S_c be a fibre of S and write $S_c = \sum_i n_i C_i$ where the C_i are irreducible and reduced fibre components. Then*

(a) $C_i \cdot S_c = 0$*;*

(b) if $D = \sum_i m_i C_i$ *for some* $m_i \in \mathbb{Z}$ *holds, then* $D^2 \leq 0$*;*

(c) equality in (b) holds if and only if D is a rational multiple of S_c.

The first place where we use the lemma is in considering the canonical divisor class in the Néron–Severi group. Talking about divisor (classes), it is useful to have the following terminology.

Definition 1.9. We call a *divisor* vertical if it is only supported on fibre components. We call a divisor *horizontal* if its support maps surjectively to C.

Theorem 1.10. *The canonical class is the class of a fibre multiple, specifically* $K_S = (2g(C) - 2 + \chi(S))F \in \mathrm{NS}(S)$ *where F is a general (i.e. a smooth) fibre.*

Proof. Using the adjunction formula, one finds $F \cdot K_S = 0$ since by Zariski's lemma we have $F^2 = 0$. Then one can assume that K_S has only vertical components (by taking a suitable representing divisor in its class in $\mathrm{NS}(S)$. By (b), we can find $K_S^2 \leq 0$. Supposedly, one can show that in fact $K_S^2 = 0$ should hold and conclude from (c) that (up to algebraic equivalence) K_S is a fibre multiple. By using Riemann–Roch and the adjunction formula (for $\overline{O}$) yields the given multiple. □

Corollary 1.11. *For each* $P \in E(K)$ *one has* $\overline{P}^2 = -\chi(S)$*.*

From this theorem, one can further conclude that if S is a K3 surface, then $g(C)$ better be 0, so elliptic K3 surfaces can only have $C = \mathbb{P}^1$.

2 Classification of non-smooth fibres

It is possible to classify all the non-smooth fibres that may occur. There are two irreducible ones, a rational curve with a simple node or one with a cusp (types I_1 and II in Kodaira's notation). This can easily be seen by using the adjunction formula $2g(S_c) - 2 = 0$, where the right-hand side follows from Lemma 1.8 and Theorem 1.10. The reducible ones consist of two infinite families (I_n, $n > 1$ and $I_n^*, n \geq 0$) and five sporadic ones (III, IV, II^*, III^* and IV^*); this involves a bit more work.[1]

Let $S_c = \sum n_i C_i$ be a reducible fibre, hence $n_i > 0$ for all i. Then one has

$$0 = K_S \cdot S_c = \sum n_i (K_S \cdot C_i) = \sum n_i (-C_i^2 + 2g(C_i) - 2).$$

By (a) one has $C_i^2 \leq -1$ and the case $C_i^2 = -1$, $g(C_i) = 0$ is excluded by assumption (2). Therefore one has $0 \geq \sum n_i (2g(C_i) - 1)$ from which one concludes $g(C_i) = 0$ for all i and hence $C_i^2 = -2$ for all i.

Now let C_i and C_j be two different components, then one has

$$0 \geq (C_i + C_j)^2 = C_i^2 + C_j^2 + 2C_i \cdot C_j = -4 + 2C_i \cdot C_j$$

and hence $0 \leq C_i \cdot C_j \leq 2$.

If $C_i \cdot C_j = 2$ holds, then finds that $C_i + C_j$ must be the whole fibre and S_c is either type II or type I_2.

If $C_i \cdot C_j$ is either 0 or 1 for each i and j, then one can consider the intersection graph where each vertex corresponds to a component C_i and there is an edge (i, j) if and only if $C_i \cdot C_j$ equals 1. This graph is simple and one may consider the associated quadratic form, which by Zariski's lemma 1.8 is negative definite. In the next step one considers the Dynkin diagrams associated to the possible quadratic forms occuring and from these passes to the fibre types. It is then only left to calculate the suitable multiplicities. This can be done stepwise using again Zariski's lemma 1.8.

Alternatively, one could write down a Weierstrass type equation for S over $K = k(C)$ (in suitable characteristics) and study the behaviour of fibres locally. This approach makes it possible to locally determine the singularity type and is known as Tate's algorithm.

Studying the non-smooth fibres in this way also yields the result of the following theorem. For this result, one needs to distinguish between non-smooth fibres of multiplicative reduction and of additive reduction. The number m_c will denote the number of irreducible components in the fibre S_c. We will also use a parameter $\delta_c \geq 0$ that measures the 'badness' of the singularity of a fibre. For a full discussion of these concepts, see [SchSh10, chapter 4].

Definition 2.1. The topological Euler number of a variety X is the number $e(X) = \sum (-1)^i \dim H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$.

[1] For pictorial representations of these types, see [SchSh10, chapter 4].

THEOREM 2.2. *The topological Euler number of a fibre is given by*

$$e(S_c) = \begin{cases} 0 & S_c \text{ smooth}, \\ m_c & S_c \text{ multiplicative}, \\ m_c + 1 & S_c \text{ additive}. \end{cases}$$

and of the surface is given by

$$e(S) = \sum_{c \in C} \left(e(S_c) + \delta_c \right).$$

From assumption (4) we conclude the following corollary.

COROLLARY 2.3. *An elliptic surface S has $\chi(S) > 0$.*

Proof. Follows from the theorem above and Noether's formula $12\chi(S) = K_S^2 + e(S)$. □

3 THE RELATION BETWEEN $\mathrm{NS}(S)$ AND $E(K)$

THEOREM 3.1. *The Néron–Severi group* $\mathrm{NS}(S)$ *is finitely generated and torsion free.*

Proof. It is a general fact that the Néron–Severi group of a complete smooth variety is finitely generated.
Consider the cycle map

$$c : \mathrm{NS}(S) \to H^2_{\text{ét}}(S, \mathbb{Q}_\ell)$$

that preserves the pairing. Here the pairing on $H^2_{\text{ét}}(S, \mathbb{Q}_\ell)$ is the cup-product pairing to $H^4_{\text{ét}}(S, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell$. The kernel of c is $\ker(c) = \{D \in \mathrm{NS}(S) : D \equiv 0\}$ where $\equiv$ denotes numerical equivalence. Therefore $\mathrm{Num}(S) = \mathrm{NS}(S)/\equiv$ injects into a finite dimensional vector space and hence is torsion free.
The proof is concluded with the following lemma. □

LEMMA 3.2. *On an elliptic surface, algebraic equivalence and numerical equivalence coincide.*

Proof. Let D be a divisor numerical equivalent to 0. Using Riemann–Roch, Serre duality and the fact $\chi(S) > 0$, one can show that either D or $K_S - D$ is effective and one can show that, up to algebraic equivalence, D is vertical and by use of Zariki's lemma in fact a fibre multiple. Intersecting with $\overline{O}$, one concludes that this multiple is zero, so D was already algebraically equivalent to 0. □

We define a sublattice T of $\mathrm{NS}(S)$ as follows:

DEFINITION 3.3. The *trivial lattice* is $T = \langle \overline{O}, F \rangle \oplus \bigoplus_{c \in C} T_c$, where T_c is the lattice generated by fibre components not meeting $\overline{O}$.

PROPOSITION 3.4. *The lattice T is of full rank*

$$\mathrm{rk}(T) = 2 + \sum_{c \in C} (m_c - 1).$$

We have the following theorem that we state without proof.

THEOREM 3.5. *There is an isomorphism*

$$\begin{aligned} E(K) &\xrightarrow{\sim} \mathrm{NS}(S)/\,\mathrm{T} \\ P &\mapsto \overline{P} \mod T. \end{aligned}$$

This theorem gives an immediate connection between the Picard number $\rho(S) = \operatorname{rk}\mathrm{NS}(S)$ and the Mordell–Weil rank of $E(K)$.

COROLLARY 3.6. *There is the equality*

$$\rho(S) = 2 + \sum_{c \in C}(m_c - 1) + \operatorname{rk} E(K).$$

References

[BHPV04] Wolf P. Barth, Klaus Hulek, Chris A.M. Peters, Antonius van de Ven, *Compact Complex Surfaces*, Springer-Verlag, Berlin Heidelberg, 2004

[SchSh10] Matthias Schütt and Tetsuji Shioda, *Elliptic surfaces*, 2010, http://arxiv.org/abs/0907.0298