Elliptic K3 surfaces

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An elliptic K3 surface is a K3 surface together with an elliptic fibration. In the first part of these notes we will give some chriterions for the existence of an elliptic fibrations, and eventually also of a section, on a K3 surface.

In the second part of the notes we will focus on the singular fibres of an elliptic K3 surfaces. We will see which kind of singular fibres and which configurations of singular fibres can occur on a K3 surface.

1 Existence of an elliptic fibration

An *elliptic K3 surface* is a K3 surface X with a surjective morphism $\pi\colon X \to \mathbb{P}^1$, such that the generic fibre of π is a smooth integral curve of genus 1.

Remark 1.1. This is equivalent to asking that there is a closed point t of $\mathbb{P}^1$ such that $X_t = \pi^{-1}(t)$ is an integral smooth curve of genus 1.

Example 1.2. *Let X be the Kummer surface associated to the abelian variety $A = E_1 \times E_2$, given by the product of two elliptic curves E_1, E_2. Then X admits (at least) two elliptic fibrations, namely*

$$\pi_i : X \to E_i/\{\pm 1\} \cong \mathbb{P}^1,$$

where $i = 1, 2$.

Lemma 1.3. *Let X be a K3 surface, then the following statements hold.*

1. *Let $\pi : X \to C$ be a surjective morphism from X onto a curve C. Then C is rational.*
2. *If $\pi\colon X \to \mathbb{P}^1$ is an elliptic fibration then not all the fibres can be smooth.*
3. *Any smooth irreducible fibre of an arbitrary surjective morphism $\pi\colon X \to \mathbb{P}^1$ is an elliptic curve.*

Proof. 1. Consider the surjective morphism $X \to C$ and its Stein factorisation $X \to \tilde{C} \to C$ (see [Har77, Corollary III.11.5]). We may assume tat $\tilde{C}$ is smooth, and since the general fibre of $X \to \tilde{C}$ is geometrically integral we have that $H^1(\tilde{C}, \mathcal{O}) \hookrightarrow H^1(X, \mathcal{O}) = 0$ we have that $\tilde{C} \cong \mathbb{P}^1$.

2. Assume all the fibres are smooth. Then, by the multiplicavity of the topological Euler number we would have

$$24 = e(X) = e(X_t) \cdot e(\mathbb{P}^1) = 0.$$

3. Let $t \in \mathbb{P}^1$ be such that X_t is a smooth fibre. Since all the points on $\mathbb{P}^1$ are linearly equivalent we have that all the fibers are linearly equivalent, and so $X_t^2 = 0$. By the adjunction formula we have

$$0 = X_t^2 = 2g(X_t) - 2,$$

and so $g(X_t) = 1$.

□

Proposition 1.4. *Let X be a K3 surface defined over a field k with characteristic 0. Let L be a non-trivial nef line bundle on X, such that $L^2 = 0$. Then L is base point free and there exists a smooth irreducible genus 1 curve $E \subset X$ such that $dE \in |L|$ for some integer $m > 0$.*

Proof. Since $L^2 = 0$, by the Riemann-Roch for K3 surfaces we know that that either L or $-L$ is effective. By assumption L is nef, and therefore $-L$ cannot be effective. It follows that that L is effective and then, using Serre duality we have that $h^2(X, L) = h^0(X, -L) = 0$. We can then conclude that $h^0(X, L) \geq 2$.

Let F be the fixed part of L, and define the mobile part $M := L - F$. By definition, M has at most fixed points. Notice that M is non-trivial, since $h^0(X, M) = h^0(X, L) \geq 2$. Also, the mobile part M is nef, $M^2 \geq 0$ and $M \cdot F \geq 0$. From the hypothesis we have that $L^2 = 0$, but $L = M + F$, and so we get

$$L \cdot M + L \cdot F = 0.$$

Now notice that L, M and F are all effective, and so $L \cdot M \geq 0$ and $L \cdot F \geq 0$. It follows that $L \cdot M = L \cdot F = 0$. And so we have that $0 = L \cdot M = (M + F) \cdot M = M^2 + F \cdot M = 0$. Also in this case we have that M^2 and $F \cdot M$ are both non-negative, yelding $M^2 = 0$ and $F \cdot M = 0$. From $F \cdot M = 0$ and $L \cdot F = 0$ it easily follows that $F^2 = 0$.

Assume now that F is non-trivial. Then, again by the Riemann-Roch for K3 surfaces, it follows that $h^0(X, F) \geq 2$, but F is the fixed part of L and hence $h^0(X, F) \geq 1$. It follows that F must be trivial and hence L has at most fixed points. But $L^2 = 0$, therefore it has no fixed points.

Let $\phi_L : X \to \mathbb{P}^m$ the map defined by the linear system $|L|$, where $m = h^0(X'L) - 1$. Since L is base point free, ϕ_L is a morphism. We claim that the image $D := \phi_L(X)$ is a curve in $\mathbb{P}^m$. Notice that the image of X has dimension either 1 or 2. Assume that it has dimension 2. Consider an hyperplane $H \subset \mathbb{P}^m$. The hyperplane section $H \cap D$ is a curve inside D, and its pullback is linearly equivalent to L. But an hyperplane section is an ample divisor, and ample divisors have positive self intersection, yelding $L^2 \geq 0$. This is contradiction with the hypothesis that $L^2 = 0$, and therefore the dimension of D must be 1.

Then, by Lemma 1.3.i), we have that D is rational.

Consider the Stein factorization

$$X \xrightarrow{\phi'_L} \tilde{D} \xrightarrow{\pi} D \, ,$$

where ϕ'_L has connected components and π is finite. Notice that, by Lemma 1.3.i), also $\tilde{D}$ is rational. Let E be the generic fibre of $\phi'_L : X \to \tilde{D}$. Since $\tilde{D}$ si rational, all its points are linearl equivalent, therefore the fibres of ϕ'_L form a linear systmen. By Bertini's theorem we know that it is smooth and, by Lemma 1.3.iii), we know that it has genus 1. Since all the fibers of ϕ' are connected it follows that E is irreducible. As $D \cong \mathbb{P}^1$, all the fibers are linearly equivalent, and in particular

$$L \sim \phi_L^*(D \cap H) = (\pi \circ \phi'_L)^*(D \cap H) = (\phi'_L)^* \pi^*(D \cap L) \sim (\deg \pi)(\deg D)E.$$

It follows that $(\deg \pi)(\deg D)E \in |L|$. □

Remark 1.5. In [Huy14, Proposition 2.3.10] it is implicitely stated that $d = m$, but I have not been able to prove that.

Proposition 1.6. *Let X be a K3 surface defined over an algebraically closed field k with charachteristic* 0*. Then the following statements hold:*

1. *X admits an elliptic fibration if and only if there is a divisor class $L \in \mathrm{NS}(X)$ such that $L^2 = 0$.*
2. *If $\rho(X) \geq 5$ then X admits an elliptic fibration.*
3. *The surface X admits at most finitely many non-isomorphic elliptic fibrations.*

Proof. 1. Notice that one implication is trivial: the existence of an elliptic fibration immediately implies the existence of a 0-class. It is left to show the converse. By Proposition 1.4 it is enough show that there exist a nef line bundle L' with $L'^2 = 0$. Since $L^2 = 0$ we have that $L \in \overline{\mathcal{C}_X}$. Without loss of generality we assume that $L \in \overline{\mathcal{C}_X^+}$. It follows that L is effective. If L is not nef then there exist a -2-curve $C \subset X$ such that $L \cdot C < 0$. Consider the reflection $s_{[C]}$ of the Néron-Severi group induced by C:

$$s_{[C]} : x \mapsto x + (x \cdot C)C.$$

Such reflection is an isometry that preserves the positive cone and therefore $s_{[C]}(L) \in \overline{\mathcal{C}_X^+}$. As before, it follows that $s_{[C]}(L)$ is effective. Moreover, $s_{[C]}(L) \cdot H = L \cdot H + (L \cdot C)(C \cdot H) < L \cdot H$, where H is a fixed hyperplane section. If $s_{[C]}(L)$ is not nef, then iterate the process. Since the intersection of an effective divisor with a hyperplane section must be positive but it decreases at each step, the process must stop. So we have a sequence of -2 curves $C_1, ..., C_k$ such that $L' = (s_{[C_k]} \circ ... \circ s_{[C_1]})(L)$ si nef and of course $L'^2 = 0$. The statement hence follows from Proposition 1.4.

2. An indefinite lattice of rank greater or equal to 5 represents 0. (See [Ser73, Corrolary IV.2.3.2])

3. There are only finitely many orbits of the action of $\mathrm{Aut}(X)$ on the set of classes $\alpha \in \mathrm{NS}(X)$ of the form $\alpha = E$ with $E \subset X$ irreducible and $\alpha^2 = E^2 = 0$. (See [Huy14, Corrolary 8.4.5])

□

Let (X, π) be an elliptic K3 surface. We say that (X, π) admits a section if there is a curve $C \subset X$ such that π induces an isomoprhism from C to $\mathbb{P}^1$.

Corollary 1.7. *Let X be a K3 surface and assume that the lattice U can be embedded into* $\mathrm{NS}(X)$. *Then X admits an elliptic fibration together with a section.*

Proof. Let e, f be the generators of the image of U inside $\mathrm{NS}(X)$. Then e can serve as elliptic fibration and $f - e$ as section. □

Corollary 1.8. *Let X be a K3 surface with Picard number $\rho(X) \geq 12$. Then X admits an elliptic fibration together with a section.*

Proof. If $\rho(X) \geq 12$, then there is an embedding $U \hookrightarrow \mathrm{NS}(X)$ (see [Huy14, Corollary 14.3.7, Corollary 14.1.18]). The statement follows from Corollary 1.7 □

2 Singular fibres

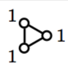
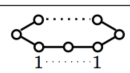
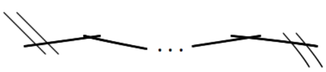
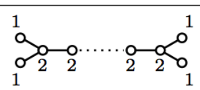
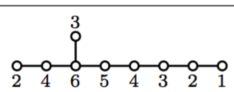
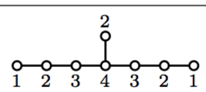
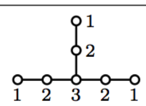
Let X be an elliptic surface K3 surface with $\pi : X \to \mathbb{P}^1$ being its elliptic fibration.

Proposition 2.1 (Zariski's lemma)**.** *If $X_t = \sum_{i=1}^{l} m_i C_i$ with C_i integral, then $C_i \cdot X_t = 0$. Moreover, $(\sum_{i=1}^{l} n_i C_i)^2 \leq 0$ for all the choices of n_i and equality holds if and only if $n_1/m_1 = \ldots = n_l/m_l$.*

Corollary 2.2. *Either a fibre X_t is irreducible (and in fact integral) or it is a curve of the form $X_t = \sum_{i=1}^{l} m_i C_i$ with $C_i \cong \mathbb{P}^1$ and $(m_1, \ldots, m_l) = 1$.*
If $l > 2$ then $C_i \cdot C_j \leq 1$ for all i, j. If $l = 2$ then $C_i \cdot C_j = 2$.

The next result shows which kind of singular fibres we can have for an elliptic fibration of a K3 surface X. It was proved by Kodaira for complex surfaces and by Néron and Tate for positive characteristic.

Theorem 2.3. *The isomorphism types of the singular fibres $X_t = \sum_{i=1}^{l} m_i C_i$ of an elliptic K3 surface $\pi : X \to \mathbb{P}^1$ are classified by the table below. The vertices are labelled by the coefficients m_i and the last column gives the topological Euler number.*

I_0	smooth elliptic 	$\widetilde{A}_0$	$e = 0$
I_1	rational curve with DP 	$\widetilde{A}_0$	$e = 1$
II	rational curve with cusp 	$\widetilde{A}_0$	$e = 2$
III		$\widetilde{A}_1$	$e = 3$
I_2		$\widetilde{A}_1$	$e = 2$
IV		$\widetilde{A}_2$ 	$e = 4$
$\mathrm{I}_{n\geq 3}$		$\widetilde{A}_{n-1}$ 	$e = n$
I_n^*		$\widetilde{D}_{n+4}$ 	$e = n + 6$
II^*		$\widetilde{E}_8$ 	$e = 10$
III^*		$\widetilde{E}_7$ 	$e = 9$
IV^*		$\widetilde{E}_6$ 	$e = 8$

Remark 2.4. The possible configurations of the singular fibers are restricted by the topology of the K3 surfaces. Infact, by the additivity of the euler number, we have that

$$24 = e(X) = \sum_{t \in \mathbb{P}^1} e(X_t).$$

All but finitely many fibers are smooth, and therefore $e(X_t) \neq 0$ for only finitely many values of t. Since $e(X_t) \geq 0$, we have only finitely many configurations.

Example 2.5. *Let X be again the Kummer surface associated to the product of elliptic curves $E_1 \times E_2$, and consider the elliptic fibration $\pi_1 \colon X \to E_1/\{\pm 1\}$ (cf. Example 1.2). The only singular fibres are the fibres above the image of the four 2-torsion points. The fibre over such a point is isomoprhic to $E_2/\{\pm 1\}$ and contains the images of four of the sixteen 2-torsion points of $E_1 \times E_2$. These points are blown up while passing to X. It follows that X has four singular fibres, all of type I_0^*.*

Remark 2.4 gives us a first necessary condition for a configuration of singular fibres on a K3 surfaces. The next results will give us more information about the possible singular fibres of a K3 surface.

Theorem 2.6. *In every characteristic $p \neq 2$, the maximal singular fibres of an elliptic K3 surface X are of type I_{19} and I_{14}^*. In characteristic 2 they are I_{18} and I_{13}^*.*

Proof. See [Sch06, Theorem 1.1]. □

Remark 2.7. The fibres I_{19} and I_{14}^* are actually realized as singular fibres on an elliptic K3 surface. See [Shi03] for an example of an elliptic K3 surface with a singular fibre of type I_{19}.

Remark 2.8. In [Shi00], Shimada gives a complete list of sublattices of ADE-type that can be realized as the sublattice generated by the reducible singular fibers of an elliptic K3 surface. There is a correspondece between the configuration of such a sublattice and the configuration of the singular fibres. The correspondence is almost a 1-to-1 correspondence, but it may happen that more than one configurations of singular fibres correspond to the same sublattice. For more details about the correspondence see [Shi00, Paragraph 2.3] The list consists of 3693 possible sublattices.

References

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