Notes on K3 surfaces

Dino Festi

1 The Picard group

Let X be a smooth irreducible projective variety over a field k.
A prime divisor on X is an irreducible closed subvariety of codimension 1 defined over k.
We define the divisor group of X, denoted by $\mathrm{Div}(X)$, as the free abelian group generated by the prime divisors:

$$\mathrm{Div}(X) := \left\{ \sum n_Z \cdot Z \mid n_Z \in \mathbb{Z},\, n_Z = 0 \ \ \forall\forall Z \right\}.$$

Remark 1.1. Prime divisors may be singular.

Remark 1.2. If the field k is not algebraically closed then prime divisors may be geometrically reducible.

Since X is a smooth variety, every prime divisor Z induces a discrete valuation $v_Z \colon k(X) \to \mathbb{Z}$ on the function field $k(X)$ of X. Using these valuations we define the map

$$\mathrm{div} \colon k(X) \to \mathrm{Div}(X)$$

by sending the function $f \in k(X)$ to the divisor $\mathrm{div}(f) := \sum v_Z(f) \cdot Z$.

Proposition 1.3. *The map* $\mathrm{div} \colon k(X) \to \mathrm{Div}(X)$ *is well defined.*

Proof. Let f be an element of the function field. Then f can be expressed as a ration of polynomials. Both numerator and denominator of f have only finitely many irreducible factors. The prime divisors corresponding to the curves defined by those factors (finitely many) are exactly those with non-zero coefficients in $\mathrm{div}(f)$. □

The elements in the image of the map div, denoted by $\mathrm{PDiv}(X)$, are called *principal divisors* of X.

Given two divisors $D_1, D_2 \in \mathrm{Div}(X)$, we say they are *linearly equivalent*, denoted by $D_1 \sim_{\mathrm{lin}} D_2$, if $D_1 - D_2$ is a principal divisor.

We define the Picard group of X as

$$\mathrm{Pic}(X) := \mathrm{Div}(X)/\sim_{\mathrm{lin}} = \mathrm{Div}(X)/\mathrm{PDiv}(X).$$

Remark 1.4. In more generality, $\mathrm{Pic}(X)$ is defined as the group of isomorphic classes of line bundles of X, with the sum being the tensor product of sheaves and the inverse being the dual. Yet the two definitions are equivalent if X is smooth.

Example 1.5. *Let $X \subseteq \mathbb{P}^n$ be a smooth variety. A hyperplane section of X is the intersection of X with an hyperplane. Any two hyperplane sections are linearly equivalent. In order to see it consider two hyperplanes H_1 and H_2. They are defined by two linear equations, so we can write $H_i \colon l_i = 0$, where l_i, $i = 1, 2$ is a linear polynomial. Then $D_1 - D_2 = \mathrm{div}(l_1/l_2) \in \mathrm{PDiv}(X)$.*

2 Intersection numbers

Let X be a smooth surface over an algebraically closed field k.
Let $C_1, C_2 \subseteq X$ be two curves on X, given by the equations $f_i = 0$, with $i = 1, 2$ respectively.
We say that C_1 and C_2 *intersects transversally* at $P \in C_1 \cap C_2$ if the maximal ideal of the local ring $\mathcal{O}_{X,P}$ is generated by f_1 and f_2.

Let D_1, D_2 be two prime divisors intersecting transversally, the we define their *intersection number* as

$$D_1 \cdot D_2 = \#(D_1 \cap D_2).$$

Proposition 2.1. *Let X be a smooth surface.*
The definition of intersection numbers extends to a symmetric bilinear pairing

$$\mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$$

that respects linear equivalence.

Proof. See [Har77, Theorem V.1.1, p. 357]. □

Corollary 2.2. *Let X be a smooth surface.*
The definition of intersection numbers extends uniquely to a symmetric bilinear pairing

$$\mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}.$$

Proof. It follows from the fact that the intersection respects the linear equivalence. □

Let $[D_1]$ and $[D_2]$ two classes of divisors in $\mathrm{Pic}(X)$. We say that $[D_1]$ and $[D_2]$ are *numerically equivalent*, denoted by $[D_1] \sim_{\mathrm{num}} [D_2]$ if $[D_1] \cdot [E] = [D_2] \cdot [E]$ for any class of divisor $[E] \in \mathrm{Pic}(X)$.
We define

$$\mathrm{Pic}^n(X) := \{[D] \in \mathrm{Pic}(X) \mid D \sim_{\mathrm{num}} 0\}$$

and

$$\mathrm{Num}(X) := \mathrm{Pic}(X)/\mathrm{Pic}^n(X).$$

From now on we will drop the notation $[D]$ to denote a class of a divisor in $\mathrm{Pic}(X)$ and we will only write D, specifying whether it is to be viewed as an element of $\mathrm{Pic}(X)$ or $\mathrm{Div}(X)$.

Remark 2.3. Let X be a smooth surface and let D an element of $\mathrm{Pic}(X)$, its self-intersection number is $D^2 = D \cdot D$.
If $X \subseteq \mathbb{P}^n$ is a projective smooth variety and $H \subseteq X$ is a hyperplane section, then H^2 is called the *degree* of X.

Example 2.4. *Let $X \subseteq \mathbb{P}^3$ be a surface defined by a polynomial of degree d, and let H be a plane section on X. Then $H^2 = d$. Indeed, if H' is another plane sections intersecting H transversally, we have that $H^2 = H \cdot H' = \#(X \cap H \cap H') = \#((H \cap X) \cap (H \cap H'))$. But $H \cap X$ is a plane curve of degree d and $H \cap H'$ is a line. Then, by Bezout's theorem, $H^2 = d$.*

Example 2.5. *If $X \subseteq \mathbb{P}^n$ is a smooth surface and $C \subset X$ is a curve on X, then $C \cdot H$ equals the degree of the curve.*

Theorem 2.6 (Adjunction formula)**.** *If C is a nonsingular curve of genus g on the surface X, and if K is the canonical divisor of X, then*

$$2g - 2 = C \cdot (C + K).$$

Proof. See [Har77, Proposition V.1.5]. □

Theorem 2.7 (Hurwitz's formula)**.** *Let $f\colon X \to Y$ be a finite covering of nonsingular curves. Let n be the degree of f, then*

$$2g(X) - 2 = n(2g(Y) - 2) + \sum_{P \in X} (e_P - 1).$$

Proof. See [Har77, Corollary IV.2.4]. □

3 Complex structure on $\mathrm{Pic}(X)$

Let X be a projective smooth complex variety, and consider the exponential map of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^\times \longrightarrow 0\,,$$

where the first map is defined by sending n to the constant function $2\pi in$, and the second map is the exponential map $f \mapsto e^f$.

The exponential sequence induces the following exact sequence of cohomology groups:

$$\cdots \longrightarrow H^1(X(\mathbb{C}), \mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{O}_X^\times) \longrightarrow H^2(X(\mathbb{C}), \mathbb{Z}) \longrightarrow \cdots,$$

Proposition 3.1. *1.* $H^1(X, \mathcal{O}_X^\times) = \mathrm{Pic}(X)$*.*

2. $H^1(X, \mathcal{O}_X)$ *is a finite dimensional* $\mathbb{C}$*-vector space and* $H^1(X(\mathbb{C}), \mathbb{Z})$ *is a lattice in it.*

3. Let $\mathrm{Pic}^0(X)$ *be he image of* $H^1(X, \mathcal{O}_X)$ *in* $\mathrm{Pic}(X)$*. Then* $\mathrm{Pic}^0(X)$ *is a torus and* $\mathrm{Pic}^0(X) \subseteq \mathrm{Pic}^n(X)$*.*

Proof. 1. See [Har77, Exercise III.4.5].

2. $H^1(X(\mathbb{C}), \mathbb{Z})$ is a $\mathbb{Z}$-module.

3. The first statement follows trivially from point *2*.
For the second statemente, just notice that the abelian group $\mathrm{Pic}^0(X)$ is the image of $\mathbb{C}$-vector space, therefore it is divisible and let N be an element of $\mathrm{Pic}^0(X)$ and let n be an integer. Then there is an element M in $\mathrm{Pic}^0(X)$ such that $N = nM$. This means that for any class $E \in \mathrm{Pic}(X)$ the intersection number $N \cdot E = nM \cdot E = n(M \cdot E)$ is divisible by n. Then, by the generality of n, it follows that $N \cdot E = 0$ for every $E \in \mathrm{Pic}(X)$.

□

By the facts above we have that we have that the image of $\mathrm{Pic}(X)$ inside $H^2(X, \mathbb{Z})$ is isomorphic to the quotient $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$. This quotient is called the *Néron-Severi* group of X, and it is denoted by $\mathrm{NS}(X)$.

Theorem 3.2 (Néron-Severi)**.** *Let X be a smooth projective variety over a field k. Then* $\mathrm{NS}(X)$ *is a finitely generated abelian group.*

From the Néron-Severi theorem it follows that

$$\mathrm{NS}(X) \cong \mathbb{Z}^\rho \oplus \mathrm{NS}_{\mathrm{tor}}(X).$$

The rank $\rho = \rho(X)$ is called the *picard number* of X.

Proposition 3.3.

$$\mathrm{Num}(X) := \mathrm{Pic}(X)/\mathrm{Pic}^n(X) \cong \mathrm{NS}(X)/\mathrm{NS}_{\mathrm{tor}}(X).$$

Proof. Let $\mathrm{Pic}^\tau(X) := \{N \in \mathrm{Pic}(X) | \exists n \in \mathbb{Z} : nN \in \mathrm{Pic}^0(X)\}$. Then we have the following chain of inclusions:

$$\mathrm{Pic}^0(X) \subseteq Pic^\tau(X) \subseteq \mathrm{Pic}^n(X) \subseteq \mathrm{Pic}(X).$$

It is a fact that $\mathrm{Pic}^\tau(X) = \mathrm{Pic}^n(X)$ see [Har73, Proposition 3.1]. But $\mathrm{Pic}^\tau(X)$ corresponds exactly to the set of torsion elements of $\mathrm{NS}(X)$. This concludes the proof. □

We can summarize all the constructions we have been through in the following diagram:

$$\begin{array}{ccccc}
\mathrm{PDiv}(X) & \hookrightarrow & \mathrm{Div}(X) & \longrightarrow & \mathrm{Pic}(X) \\
\downarrow & & \downarrow & & \downarrow \\
\{0\} & \hookrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{Pic}(X) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Pic}^0(X) & \hookrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{NS}(X) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Pic}^n(X) & \hookrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{Num}(X)
\end{array}$$

Proposition 3.4. *Let X be a complex algebraic K3 surface. Then:*

1. $H^1(X(\mathbb{C}), \mathbb{Z}) = 0$ *and* $H^2(X(\mathbb{C}), \mathbb{Z}) \cong \Lambda := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$*;*

2. *the natural surjections*

$$\mathrm{Pic}(X) \longrightarrow \mathrm{NS}(X) \longrightarrow \mathrm{Num}(X)$$

are isomorphisms.

Proof. 1. See [BHPVdV04, Proposition VIII.3.3].

2. Recall that $\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^0(X)$ is the image of $H^1(X, \mathcal{O}_X)$ in $\mathrm{Pic}(X)$, but for K3 surfaces $H^1(X, \mathcal{O}_X) = 0$. This proves that the first map is an isomorphism.
The second statement follows from the fact that $H^2(X(\mathbb{C}), \mathbb{Z})$ is free (see the first statement of this proposition), and from Proposition 3.3. □

Remark 3.5. Using Proposition 3.4 and Proposition 3.3 it follows that, if X is a K3 surface, the sequence on the cohomology groups becomes

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathrm{NS}(X) \longrightarrow \Lambda \longrightarrow \cdots,$$

hence the Néron-Severi group can be viewed as a sublattice of Λ.

Remark 3.6. The isomorphism $H^2(X(\mathbb{C}), \mathbb{Z}) \cong \Lambda$ is not unique. The choice of such an isomorphism is called a *marking* of the K3 surface.

Theorem 3.7 (Hodge index)**.** *Let X be a smooth surface. Let H be an ample divisor on X and suppose that D is a divisor not numerically equivalent to* 0 *and such that $D \cdot H = 0$. Then $D^2 < 0$.*

Proof. See [Har77, Theorem V.1.9]. □

Corollary 3.8. *The signature of the intersection form on* $\mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ *is* $(1, \rho - 1)$.

Proof. It follows from Sylvester's law of inertia. □

Theorem 3.9 (Lefschetz)**.** *Let X be compact surface. Then* $\mathrm{NS}(X) \cong H^{1,1}(X) \cap H^2(X, \mathbb{Z})$.

Proof. See [BHPVdV04, Theorem IV.2.13]. □

Remark 3.10. From Proposition 3.4 it easily follows that the picard number ρ of a K3 surface is always less or equal to 22.
Recalling the Hodge decomposition for the $H^2(X)$ of a K3 surface, Theorem 3.9 tells us that infact $\rho \leq 20$.

References

[BHPVdV04] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 2004.

[Har73] Robin Hartshorne. Equivalence relations on algebraic cycles and subvarieties of small codimension. In *Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974)*, pages 129–164. Amer. Math. Soc., Providence, R.I., 1973.

[Har77] R. Hartshorne. *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.