Ample and Nef cones

Giulio Orecchia

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1 Notation

In what follows, a *variety* will always denote a separated, geometrically integral scheme of finite type over a field k.

Throughout the lecture, a *surface* will mean a variety of dimension 2, while a *curve* on it will mean a prime divisor (i.e. a closed subvariety of dimension 1).

2 Line bundles

In the last lecture, the group $\operatorname{Pic}(X)$ was defined for X a smooth projective variety over a field k as the quotient $\operatorname{Div}(X)/\operatorname{PDiv}(X)$. It turns out that the group $\operatorname{Pic}(X)$ is isomorphic to the group of isomorphism classes of line bundles on X, with operation given by the tensor product and with inverse $\mathcal{L}^{-1} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$. See Hartshorne's book for a detailed explanation. Therefore we can define an intersection form on isomorphism classes of line bundles on X by $(\mathcal{L}_1.\mathcal{L}_2) := (D_1.D_2)$ for $[D_1], [D_2]$ the divisor classes corresponding to $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively.

3 Ample line bundles

Definition 3.1. Let Y be a scheme and X a scheme over Y. We say that a line bundle $\mathcal{L}$ on X is *very ample* relative to Y if for some $r \in \mathbb{Z}_{\geq 0}$ there exists an immersion (i.e. an open immersion followed by a closed immersion) of Y-schemes $i : X \to \mathbb{P}^r_Y$ such that $\mathcal{L} \cong i^*(\mathcal{O}(1))$.

Remark 3.2. Let Y be noetherian, X proper over Y and $\mathcal{L}$ a very ample line bundle on X relative to Y isomorphic to $i^*\mathcal{O}(1)$ with $i : X \to \mathbb{P}^r_Y$ an immersion. Then i is a closed immersion. Indeed, by [1, Ex. II.4.4.], the image of the immersion i is closed in $\mathbb{P}^r_Y$.

Remark 3.3. In the case of $Y \cong \operatorname{Spec} A$, saying that $\mathcal{L}$ is very ample is equivalent to asking that $\mathcal{L}$ is generated by global sections $s_0, \ldots, s_r$ such that the corresponding Y-morphism $X \to \mathbb{P}^r_A$ is an immersion. This is an immediate consequence of [1, Theorem II.7.1.]

Definition 3.4. Let X be a noetherian scheme. A line bundle $\mathcal{L}$ on X is called *ample* if for every coherent sheaf $\mathcal{F}$ on X there is an integer n such that for every $m \geq n$ the sheaf $\mathcal{F} \otimes \mathcal{L}^m$ is generated by its global sections.

Remark 3.5. If X is affine, every invertible sheaf is ample, since every coherent sheaf is the sheaf associated to a finitely generated $\Gamma(X, \mathcal{O}_X)$-module.

Theorem 3.6. Let X be a scheme of finite type over a noetherian ring A and let $\mathcal{L}$ be a line bundle on X. Then $\mathcal{L}$ is ample if and only if there exists an integer $n_0 > 0$ such that $\mathcal{L}^m$ is very ample over $\operatorname{Spec} A$ for all $m \geq n_0$.

Proof on [1, Theorem II.7.6.] plus [1, Ex. II.7.5.]

Let's see some facts about line bundles. Let $\mathcal{L}$ and $\mathcal{M}$ be line bundles on a scheme X of finite type over a noetherian ring A.

i) If $\mathcal{L}$ is ample and $\mathcal{M}$ is arbitrary, then $\mathcal{L}^n \otimes \mathcal{M}$ is ample for sufficiently large n. This follows immediately by the fact that if $\mathcal{L}$ is ample and $\mathcal{M}$ is generated by global sections, $\mathcal{L} \otimes \mathcal{M}$ is ample.

ii) If $\mathcal{L}$ and $\mathcal{M}$ are two (very) ample line bundles, then so is $\mathcal{L} \otimes \mathcal{M}$. This follows from the fact that the Segre embedding $\mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{(r+1)(s+1)-1}$, $(a_0, \ldots, a_r), (b_0, \ldots, b_s) \mapsto (\ldots, a_i b_j, \ldots)$ in lexicographic order, is an immersion.

iii) Let X be a geometrically integral proper scheme over a field k. If both $\mathcal{L}$ and its dual $\mathcal{L}^{-1}$ are ample, then $\mathcal{L}$ is torsion in $\operatorname{Pic}(X)$ and $X = \operatorname{Spec} k$. Indeed, for some $m > 0$ both $\mathcal{L}^m$ and $\mathcal{L}^m$ are very ample and hence generated by global sections. Therefore there are non-zero morphisms

$$\varphi : \mathcal{O}_X \to \mathcal{L}^m \text{ global section of } \mathcal{L}^m$$

and

$$\psi : \mathcal{L}^m \to \mathcal{O}_X \text{ global section of } \mathcal{L}^{-m}.$$

By composing we get two morphisms $\mathcal{L}^m \to \mathcal{L}^m$ and $\mathcal{O}_X \to \mathcal{O}_X$ which are both non-zero by integrality of X. Then they are isomorphisms since $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{L}^m, \mathcal{L}^m) = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = k$. Hence φ is an isomorphism. Now, if $\mathcal{O}_X$ is very ample, X embeds into $\mathbb{P}^0_k = \operatorname{Spec} k$. So $X = \operatorname{Spec} k$ since $\Gamma(X, \mathcal{O}_X) = k$.

Let X be a smooth projective variety over a field k.

Definition 3.7. We say that a divisor D on X is ample, resp. very ample, if the associated line bundle $\mathcal{O}(D)$ is so.

Definition 3.8. A divisor $D = \sum n_i D_i$ on X is *effective* if $n_i \geq 0$ for all i.

Remark 3.9. Every very ample divisor is linearly equivalent to an effective one. Indeed, if D is very ample and $\mathcal{O}(D)$ is the corresponding very ample line bundle, then $\Gamma(X, \mathcal{O}(D)) \neq 0$. If $s \in \Gamma(X, \mathcal{O}(D))$ is a non-zero global section, we can associate to it a divisor D' in the following way: for every prime divisor Z, we let s_Z be a generator of $\mathcal{O}(D)_{\eta_Z}$, where η_Z is the generic point of Z. Then there is a unique $a \in \mathcal{O}_{\eta_Z}$ such that $a s_Z = s$. If we let n_Z be the valuation of a, then $D' = \sum n_Z Z$ is effective and linearly equivalent to D.
One can also recover D' as the schematic support of the cokernel of the map

$$\mathcal{O} \to \mathcal{O}(D)$$

given by the global section s.

Recall that for X a smooth projective surface over a field, there exists a bilinear symmetric form (.) on $\operatorname{Div}(X)$ which extends uniquely to $\operatorname{Pic}(X)$. The following theorem plays a central role in this lecture.

Theorem 3.10 (**Nakai-Moishezon-Kleiman**)**.** A line bundle $\mathcal{L}$ on a smooth projective surface X over a field k is ample if and only if $(\mathcal{L})^2 > 0$ and $(\mathcal{L}.\mathcal{O}(C)) > 0$ for all curves $C \subset X$.

For a proof see [1, Theorem V.1.10.]

As a consequence, the notion of ampleness is invariant under numerical equivalence. Therefore we can speak about ample classes in the groups $\mathrm{Num}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^n(X)$ and $\mathrm{NS}(X) = \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ (since $\mathrm{Pic}^0(X) \subset \mathrm{Pic}^n(X)$).

It is also natural to define in $\mathrm{Num}(X)$ another class of line bundles.

Definition 3.11. A *nef* line bundle on a smooth projective surface is a line bundle $\mathcal{L}$ such that $(\mathcal{L}.\mathcal{O}(C)) \geq 0$ for all curves $C \subset X$.

4 Riemann-Roch on surfaces

Theorem 4.1 (Riemann-Roch). Let X be a smooth projective surface over a field k. Let $\mathcal{L}$ be a line bundle on X. Then

$$\chi(X, \mathcal{L})) = 1/2(\mathcal{L}.\mathcal{L} + \mathcal{L}.\omega_{X/k}^{-1}) + \chi(X, \mathcal{O}_X).$$

5 Ample and nef cone

A subset $\mathcal{C} \subset V$ of a real vector space V is a *cone* if $\mathbb{R}_{\geq 0} \cdot \mathcal{C} = \mathcal{C}$.
Let X be a smooth projective surface. The form (.) extends to the ρ-dimensional real vector space $\mathrm{NS}(X)_{\mathbb{R}}$.

Definition 5.1. The *positive cone*

$$\mathcal{C}_X^+ \subset \mathrm{NS}_{\mathbb{R}}(X)$$

is the connected component of the set $\mathcal{C}_X := \{\alpha \in \mathrm{NS}(X) | \alpha^2 > 0\}$ that contains ample classes.

Remark 5.2.

i) Of course, by Nakai-Moishezon-Kleiman Theorem, all ample classes are contained in $\mathcal{C}_X$. Notice that if H and H' are two ample classes, then for all $\alpha, \beta > 0$ we have $(\alpha H + \beta H')^2 > 0$, since $H^2 > 0, H'^2 > 0$ and $H \cdot H' \geq 0$. Indeed, for some $m > 0$, the classes mH and mH' contain very ample line bundles, hence effective divisors, and $H \cdot H' = 1/m^2(mH \cdot mH') > 0$. Therefore H and H' belong to the same connected component of the aforementioned set $\mathcal{C}_X$.

ii) The cone $\mathcal{C}_X$ has exactly two connected components, switched by multiplication by -1. By Hodge Index Theorem, we can let $e_0, e_1, \ldots, e_\rho$ be a orthonormal basis of $NS(X)_{\mathbb{R}}$ for which $e_0^2 = 1$ and $e_i^2 = -1$ for $i > 0$. If $\alpha = \sum x_i e_i$, then $\alpha^2 > 0$ implies $x_0^2 > 0$. Then
$$\mathcal{C}_X^+ = \{\alpha \in \mathcal{C}_X : x_0 > 0\}.$$

iii) If $x \in \overline{\mathcal{C}^+}_X$ is non-zero and $y \in \mathcal{C}_X$, then $y \in \mathcal{C}_X^+$ if and only if $(x.y) > 0$. To see this, consider the continuous map

$$\varphi : \mathcal{C}_X \to \mathbb{R}, \ \ y \to (x.y)$$

Assume $\varphi(y) = 0$. If $x = (x_0, \ldots, x_n)$ for a basis as in ii), and $y = (y_0, \ldots, y_n)$, then $x_0 y_0 = \sum_{i>0} x_i y_i$. By assumption, $x_0^2 \geq \sum_{i>0} x_i^2$ and $y_0^2 > \sum_{i>0} y_i^2$.. By Cauchy-Schwarz, we get

$$(x_0 y_0)^2 = (\sum_{i>0} x_i y_i)^2 \leq (\sum_{i>0} x_i^2)(\sum_{i>0} y_i^2) < x_0^2 y_0^2$$

which is a contradiction. Hence $\varphi(\mathcal{C}_X) \subset \mathbb{R} \setminus \{0\}$. Then e_0, $-e_0$ map to $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$ respectively. Hence φ is positive on $\mathcal{C}^+(X)$ and negative on $-\mathcal{C}^+(X)$.

Definition 5.3. The *ample cone*

$$\mathrm{Amp}(X) \subset \mathrm{NS}(X)_{\mathbb{R}}$$

is the set of all finite sums $\sum a_i \mathcal{L}_i$ with $\mathcal{L}_i$ ample and $a_i > 0$.
The *nef cone*

$$\mathrm{Nef}(X) \subset \mathrm{NS}(X)_{\mathbb{R}}$$

is the set of all classes $\alpha \in \mathrm{NS}(X)_{\mathbb{R}}$ with $(\alpha.C) \geq 0$ for all curves $C \subset X$.

Remark 5.4.

i) Clearly the ample cone is contained in the positive cone.

ii) It is not true in general that the nef cone is spanned by nef classes of line bundles;

iii) It is true that the two inequalities $(\alpha.\alpha) > 0$ and $(\alpha.C) > 0$ describe the ample cone.

Proposition 5.5. Let X be a smooth projective surface. If $\alpha \in \mathrm{NS}(X)$ is such that $(\alpha.C) \geq 0$ for all curves C then $\alpha^2 \geq 0$. Hence

$$\mathrm{Nef}(X) \subset \overline{\mathcal{C}^+}(X).$$

Proof. Suppose by contradiction that $\alpha^2 < 0$. By the Hodge Index Theorem,the hyperplane $\alpha^\perp$ cuts the positive cone $\mathcal{C}_X^+$ into two parts, according to the sign of $(\mathcal{L}.\alpha)$. Take $\mathcal{L} \in \mathcal{C}_X^+$ such that $(\mathcal{L}.\alpha) < 0$. Then by HAG, Corollary V.1.8, $\mathcal{L}^n$ is effective for some $n > 0$. Since α is nef, $n(\alpha.\mathcal{L}) = (\alpha.\mathcal{L}^n) \geq 0$, which is a contradiction. □

Proposition 5.6. Let X be a smooth projective surface over a field k. Then

$$\mathrm{Amp}(X) = \mathrm{Int}\,\mathrm{Nef}(X) \subset \mathrm{Nef}(X) = \overline{\mathrm{Amp}}(X)$$

where "Int" stands for "interior"

Proof. The nef cone is closed by definition, so it is enough to show that $\mathrm{Amp}(X) = \mathrm{Int}\,\mathrm{Nef}(X)$. The ample cone is open. Indeed, if H is an ample class, then $(H.H) \geq 1$ and $(H.C) \geq 1$ for any curve C. Obviously $\mathrm{Amp}(X) \subset \mathrm{Nef}(X)$, therefore $\mathrm{Amp}(X) \subset \mathrm{Int}\,\mathrm{Nef}(X)$. On the other hand, if $\alpha \in \mathrm{Int}\,\mathrm{Nef}(X)$, then $\alpha - \epsilon H$ is still in the nef cone for some $\epsilon > 0$ not too big. Then $\alpha = \alpha - \epsilon H + \epsilon H$ is a sum of a nef and an ample class, which is easily checked to be ample (a consequence of Proposition 5.5 and Remark 5.4, iii)) □

We consider now the case where X is a K_3 surface over a field k. Then the adjunction formula reads

$$(C.C) = 2g(C) - 2$$

for all curves $C \subset X$. Therefore either $(C.C) = -2$, which happens when $C \cong \mathbb{P}_k^1$, or $(C.C) \geq 0$.

The following is a corollary of Nakai-Moishezan-Kleiman Theorem.

Corollary 5.7. A line bundle $\mathcal{L}$ on a K_3 surface X over a field k is ample if and only if

i) $(\mathcal{L})^2 > 0$,

ii) $(\mathcal{L}.C) > 0$ for every smooth rational curve $\mathbb{P}^1 \cong C \subset X$, and

iii) $(\mathcal{L}.\mathcal{H}) > 0$ for one ample line bundle $\mathcal{H}$ (or, equivalently, for all of them).

Proof. If $\mathcal{L}$ is ample, then $i), ii), iii)$ are satisfied. On the other hand, for every non-rational curve on X, $(C.C) \geq 0$. Moreover, $(C.H) > 0$ for any H ample, hence $C \in \overline{\mathcal{C}^+}_X$. By i) and iii) it also follows that $\mathcal{L} \in \mathcal{C}_X^+$. Now, by Remark 5.2 iii) it follows that $(\mathcal{L}.C) > 0$. Therefore $(\mathcal{L}.C) > 0$ for all curves $C \subset X$, and we conclude by Nakai-Moishezon-Kleiman Theorem. $\square$

Corollary 5.8. For a K_3 surface over a field k one has

$$\mathrm{Amp}(X) = \{\alpha \in \mathcal{C}_X^+ : (\alpha.C) > 0 \text{ for all } \mathbb{P}^1 \cong C \subset X\}.$$

6 Chambers and walls

Let X be a K_3 surface over a field k. We have seen that for a curve $C \subset X$, $(C.C) \geq -2$ and $(C.C) = -2$ if and only if C is smooth and rational. We define the subset of -2-classes of the Neron-Severi group $\mathrm{NS}(X)$ (which, for a $K3$ surface is isomorphic to $\mathrm{Num}(X)$ and $\mathrm{Pic}(X)$):

$$\Delta := \{\delta \in \mathrm{NS}(X) : \delta^2 = -2\}$$

Let $\mathcal{L}$ be the class of a line bundle in Δ. By Riemann-Roch theorem we have

$$\chi(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L}) + h^2(\mathcal{L}) = \chi(\mathcal{O}_X) + (\mathcal{L})^2/2 + (\mathcal{L}.\omega_{X/k}^{-1})/2 = 2 + (\mathcal{L})^2/2.$$

Serre duality and triviality of $\omega_{X/k}$ imply that

$$h^0(\mathcal{L}) + h^0(\mathcal{L}^{-1}) \geq 2 + (\mathcal{L})^2/2 = 1.$$

Hence exactly one of $\mathcal{L}$ and its dual is effective. Notice also that, since for a $K3$-surface $\mathrm{NS}(X) = \mathrm{Pic}(X)$, effective divisor classes are well defined in $\mathrm{NS}(X)$.
Then we let $\Delta = \Delta^+ \cup \Delta^-$ where

$$\Delta^+ := \{\delta \in \mathrm{NS}(X) : \delta^2 = -2 \text{ and } \delta \text{ is an effective divisor class}\}, \quad \Delta^- := -\Delta^+.$$

We let $\delta^\perp \subset \mathrm{NS}(X)_\mathbb{R}$ be the hyperplane orthogonal to $\delta \in \Delta^+$.

Definition 6.1. The connected components of

$$\mathcal{C}_X^+ \setminus \bigcup_{\delta \in \Delta^+} \delta^\perp$$

are called the *chambers* of $\mathcal{C}_X^+$, while the $\delta^\perp$ are the *walls* of the chambers.

Remark 6.2. The union of the walls is closed in $\mathrm{NS}(X)_\mathbb{R}$, and the collection of the walls is locally finite in $\mathrm{NS}(X)_\mathbb{R}$. [2, Chapt. 8, Remark 2.2.]

Proposition 6.3. The ample cone $\mathrm{Amp}(X)$ is a chamber of $\mathcal{C}_X^+$.

Proof. Let

$$\aleph := \{\alpha \in \mathcal{C}_X^+ : (\alpha.\delta) > 0 \text{ for all } \delta \in \Delta^+\}.$$

This is a chamber of $\mathcal{C}_X^+$. Classes α in $\mathrm{Amp}(X)$ are given by the conditions $\alpha \in \mathcal{C}_X^+$ and $(\alpha.C) > 0$ for all curves in X with $C.C = -2$. Hence $\aleph \subset \mathrm{Amp}(X)$. On the other hand, if $H \in \mathrm{Amp}(X)$, then $(H.\delta) > 0$ for all $\delta \in \Delta^+$ since δ is an effective divisor class. So $\mathrm{Amp}(X) = \aleph$, which is a chamber of $\mathcal{C}_X^+$. □

Now, for all $\delta \in \Delta$ consider the isometry on $\mathrm{NS}(X)_{\mathbb{R}}$ given by

$$s_\delta(x) = x + (x.\delta)\delta.$$

Then $s_\delta(\delta) = -\delta$ and $s_\delta = \mathrm{id}$ on $\delta^\perp$. If $x \in \mathcal{C}_X^+$, then $s_\delta(x)^2 > 0$. Moreover $(s_\delta(x).x) > 0$, hence $s_\delta(x) \in \mathcal{C}_X^+$. Therefore the s_δ's preserve the positive cone.

Definition 6.4. The subgroup W of the orthogonal group $O(\mathrm{NS}(X)_{\mathbb{R}})$ generated by $\{s_\delta : \delta \in \Delta\}$ is called Weyl group.

The Weyl group W preserves the union of the walls $\bigcup_{\delta \in \Delta^+} \delta^\perp$. Indeed one can check that $W(\Delta) = \Delta$, and if $x \in \delta^\perp$, then $s_{\delta'}(x) \in s_{\delta'}(\delta)^\perp$ for all $\delta' \in \Delta$. Therefore W acts on the set of chambers of $\mathcal{C}_X^+$.

Proposition 6.5. The Weyl group acts simply transitively on the set of chambers of $\mathcal{C}_X^+$. The cone $\mathrm{Nef}(X) \cap \mathcal{C}^+(X)$ is a fundamental domain for the action of the Weyl group on $\mathcal{C}^+(X)$.

Proof on [2, Chapt. 8, Prop. 2.6. and Corollary 2.11]

References

[1] Robin Hartshorne, *Algebraic Geometry*
Berlin, New York: Springer-Verlag, 1977

[2] Daniel Huybrechts *Lectures on K3 surfaces*