Ample and Nef cones

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1 Notation

In what follows, a variety will always denote a separated, geometrically integral scheme of finite type over a field $k$.

Throughout the lecture, a surface will mean a variety of dimension 2, while a curve on it will mean a prime divisor (i.e. a closed subvariety of dimension 1).

2 Line bundles

In the last lecture, the group $\text{Pic}(X)$ was defined for $X$ a smooth projective variety over a field $k$ as the quotient $\text{Div}(X)/\text{PDiv}(X)$. It turns out that the group $\text{Pic}(X)$ is isomorphic to the group of isomorphism classes of line bundles on $X$, with operation given by the tensor product and with inverse $L^{-1} := \text{Hom}_{\mathcal{O}_X}(L, \mathcal{O}_X)$. See Hartshorne’s book for a detailed explanation.

Therefore we can define an intersection form on isomorphism classes of line bundles on $X$ by $(L_1, L_2) := (D_1, D_2)$ for $[D_1], [D_2]$ the divisor classes corresponding to $L_1$ and $L_2$ respectively.

3 Ample line bundles

**Definition 3.1.** Let $Y$ be a scheme and $X$ a scheme over $Y$. We say that a line bundle $L$ on $X$ is very ample relative to $Y$ if for some $r \in \mathbb{Z}_{\geq 0}$ there exists an immersion (i.e. an open immersion followed by a closed immersion) of $Y$-schemes $i : X \to \mathbb{P}^r_Y$ such that $L \cong i^*(\mathcal{O}(1))$.

**Remark 3.2.** Let $Y$ be noetherian, $X$ proper over $Y$ and $L$ a very ample line bundle on $X$ relative to $Y$ isomorphic to $i^* \mathcal{O}(1)$ with $i : X \to \mathbb{P}^r_Y$ an immersion. Then $i$ is a closed immersion. Indeed, by [1, Ex. II.4.4.], the image of the immersion $i$ is closed in $\mathbb{P}^r_Y$.

**Remark 3.3.** In the case of $Y \cong \text{Spec} A$, saying that $L$ is very ample is equivalent to asking that $L$ is generated by global sections $s_0, \ldots, s_r$ such that the corresponding $Y$-morphism $X \to \mathbb{P}^r_A$ is an immersion. This is an immediate consequence of [1, Theorem II.7.1.]

**Definition 3.4.** Let $X$ be a noetherian scheme. A line bundle $L$ on $X$ is called ample if for every coherent sheaf $\mathcal{F}$ on $X$ there is an integer $n$ such that for every $m \geq n$ the sheaf $\mathcal{F} \otimes L^m$ is generated by its global sections.

**Remark 3.5.** If $X$ is affine, every invertible sheaf is ample, since every coherent sheaf is the sheaf associated to a finitely generated $\Gamma(X, \mathcal{O}_X)$-module.
Theorem 3.6. Let $X$ be a scheme of finite type over a noetherian ring $A$ and let $L$ be a line bundle on $X$. Then $L$ is ample if and only if there exists an integer $n_0 > 0$ such that $L^n$ is very ample over $\text{Spec } A$ for all $m \geq n_0$.

Proof on [1, Theorem II.7.6.] plus [1, Ex. II.7.5.]

Let’s see some facts about line bundles. Let $L$ and $M$ be line bundles on a scheme $X$ of finite type over a noetherian ring $A$.

i) If $L$ is ample and $M$ is arbitrary, then $L^n \otimes M$ is ample for sufficiently large $n$. This follows immediately by the fact that if $L$ is ample, then $L$ is generated by global sections, $L \otimes M$ is ample.

ii) If $L$ and $M$ are two (very) ample line bundles, then so is $L \otimes M$. This follows from the fact that the Segre embedding $\mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{(r+1)(s+1)-1}$, $(a_0, \ldots, a_r), (b_0, \ldots, b_s) \mapsto (\ldots, a_i b_j, \ldots)$ in lexicographic order, is an immersion.

iii) Let $X$ be a geometrically integral proper scheme over a field $k$. If both $L$ and its dual $L^{-1}$ are ample, then $L$ is torsion in $\text{Pic}(X)$ and $X = \text{Spec } k$. Indeed, for some $m > 0$ both $L^m$ and $L^{-m}$ are very ample and hence generated by global sections. Therefore there are non-zero morphisms

$$\varphi : \mathcal{O}_X \to L^m \text{ global section of } L^m$$

and

$$\psi : L^m \to \mathcal{O}_X \text{ global section of } L^{-m}.$$ 

By composing we get two morphisms $L^m \to L^n$ and $\mathcal{O}_X \to \mathcal{O}_X$ which are both non-zero by integrality of $X$. Then they are isomorphisms since $\text{Hom}_{\mathcal{O}_X}(L^m, L^n) = \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = k$. Hence $\varphi$ is an isomorphism. Now, if $\mathcal{O}_X$ is very ample, $X$ embeds into $\mathbb{P}^n_k = \text{Spec } k$. So $X = \text{Spec } k$ since $\Gamma(X, \mathcal{O}_X) = k$.

Let $X$ be a smooth projective variety over a field $k$.

Definition 3.7. We say that a divisor $D$ on $X$ is ample, resp. very ample, if the associated line bundle $\mathcal{O}(D)$ is so.

Definition 3.8. A divisor $D = \sum n_i D_i$ on $X$ is effective if $n_i \geq 0$ for all $i$.

Remark 3.9. Every very ample divisor is linearly equivalent to an effective one. Indeed, if $D$ is very ample and $\mathcal{O}(D)$ is the corresponding very ample line bundle, then $\Gamma(X, \mathcal{O}(D)) \neq 0$. If $s \in \Gamma(X, \mathcal{O}(D))$ is a non-zero global section, we can associate to it a divisor $D'$ in the following way: for every prime divisor $Z$, we let $s_Z$ be a generator of $\mathcal{O}(D)_{\eta_Z}$, where $\eta_Z$ is the generic point of $Z$. Then there is a unique $a \in \mathcal{O}_{\eta_Z}$ such that $as_Z = s$. If we let $n_Z$ be the valuation of $a$, then $D' = \sum n_Z Z$ is effective and linearly equivalent to $D$.

One can also recover $D'$ as the schematic support of the cokernel of the map

$$\mathcal{O} \to \mathcal{O}(D)$$

given by the global section $s$.

Recall that for $X$ a smooth projective surface over a field, there exists a bilinear symmetric form $(\cdot, \cdot)$ on $\text{Div}(X)$ which extends uniquely to $\text{Pic}(X)$. The following theorem plays a central role in this lecture.

Theorem 3.10 (Nakai-Moishezon-Kleiman). A line bundle $L$ on a smooth projective surface $X$ over a field $k$ is ample if and only if $(L)^2 > 0$ and $(L, \mathcal{O}(C)) > 0$ for all curves $C \subset X$. 

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For a proof see [1, Theorem V.1.10.]

As a consequence, the notion of ampleness is invariant under numerical equivalence. Therefore we can speak about ample classes in the groups $\text{Num}(X) = \text{Pic}(X)/\text{Pic}^n(X)$ and $\text{NS}(X) = \text{Pic}(X)/\text{Pic}^0(X)$ (since $\text{Pic}^0(X) \subset \text{Pic}^n(X)$).

It is also natural to define in $\text{Num}(X)$ another class of line bundles.

**Definition 3.11.** A nef line bundle on a smooth projective surface is a line bundle $L$ such that $(L, \mathcal{O}(C)) \geq 0$ for all curves $C \subset X$.

### 4 Riemann-Roch on surfaces

**Theorem 4.1 (Riemann-Roch).** Let $X$ be a smooth projective surface over a field $k$. Let $L$ be a line bundle on $X$. Then

$$\chi(X, L) = 1/2(L.L + L.w^{-1}_X/k) + \chi(X, \mathcal{O}_X).$$

### 5 Ample and nef cone

A subset $C \subset V$ of a real vector space $V$ is a **cone** if $\mathbb{R}_{\geq 0} \cdot C = C$.

Let $X$ be a smooth projective surface. The form $(\cdot, \cdot)$ extends to the $\rho$-dimensional real vector space $\text{NS}(X)_{\mathbb{R}}$.

**Definition 5.1.** The **positive cone**

$$C^+_X \subset \text{NS}_{\mathbb{R}}(X)$$

is the connected component of the set $C_X := \{\alpha \in \text{NS}(X)|\alpha^2 > 0\}$ that contains ample classes.

**Remark 5.2.**

i) Of course, by Nakai-Moishezon-Kleiman Theorem, all ample classes are contained in $C_X$. Notice that if $H$ and $H'$ are two ample classes, then for all $\alpha, \beta > 0$ we have $(\alpha H + \beta H')^2 > 0$, since $H^2 > 0$, $H'^2 > 0$ and $H \cdot H' \geq 0$. Indeed, for some $m > 0$, the classes $mH$ and $mH'$ contain very ample line bundles, hence effective divisors, and $H \cdot H' = 1/m^2(mH \cdot mH') > 0$. Therefore $H$ and $H'$ belong to the same connected component of the aforementioned set $C_X$.

ii) The cone $C_X^+$ has exactly two connected components, switched by multiplication by $-1$.

By Hodge Index Theorem, we can let $e_0, e_1, \ldots, e_{\rho}$ be an orthonormal basis of $\text{NS}(X)_{\mathbb{R}}$ for which $e_0^2 = 1$ and $e_i^2 = -1$ for $i > 0$. If $\alpha = \sum x_i e_i$, then $\alpha^2 > 0$ implies $x_0^2 > 0$. Then

$$C^+_X = \{\alpha \in C_X : x_0 > 0\}.$$ 

iii) If $x \in \overline{C^+X}$ is non-zero and $y \in C_X$, then $y \in C^+_X$ if and only if $(x, y) > 0$. To see this, consider the continuous map

$$\varphi : C_X \to \mathbb{R}, \ y \to (x, y).$$
Assume $\varphi(y) = 0$. If $x = (x_0, \ldots, x_n)$ for a basis as in ii), and $y = (y_0, \ldots, y_n)$, then $x_0 y_0 = \sum_{i>0} x_i y_i$. By assumption, $x_0^2 \geq \sum_{i>0} x_i^2$ and $y_0^2 > \sum_{i>0} y_i^2$. By Cauchy-Schwarz, we get

$$(x_0 y_0)^2 = \left(\sum_{i>0} x_i y_i\right)^2 \leq \left(\sum_{i>0} x_i^2\right)\left(\sum_{i>0} y_i^2\right) < x_0^2 y_0^2$$

which is a contradiction. Hence $\varphi(C_X) \subset \mathbb{R} \setminus \{0\}$. Then $e_0, -e_0$ map to $\mathbb{R}_{>0}$ and $\mathbb{R}_{<0}$ respectively. Hence $\varphi$ is positive on $C^+(X)$ and negative on $-C^+(X)$.

**Definition 5.3.** The ample cone

$$\text{Amp}(X) \subset \text{NS}(X)_\mathbb{R}$$

is the set of all finite sums $\sum a_i \mathcal{L}_i$ with $\mathcal{L}_i$ ample and $a_i > 0$.

The nef cone

$$\text{Nef}(X) \subset \text{NS}(X)_\mathbb{R}$$

is the set of all classes $\alpha \in \text{NS}(X)_\mathbb{R}$ with $\langle \alpha, C \rangle \geq 0$ for all curves $C \subset X$.

**Remark 5.4.**

i) Clearly the ample cone is contained in the positive cone.

ii) It is not true in general that the nef cone is spanned by nef classes of line bundles;

iii) It is true that the two inequalities $\langle \alpha, \alpha \rangle > 0$ and $\langle \alpha, C \rangle > 0$ describe the ample cone.

**Proposition 5.5.** Let $X$ be a smooth projective surface. If $\alpha \in \text{NS}(X)$ is such that $\langle \alpha, C \rangle \geq 0$ for all curves $C$ then $\alpha^2 \geq 0$. Hence

$$\text{Nef}(X) \subset C^+(X).$$

**Proof.** Suppose by contradiction that $\alpha^2 < 0$. By the Hodge Index Theorem, the hyperplane $\alpha^\perp$ cuts the positive cone $C^+_X$ into two parts, according to the sign of $\langle \mathcal{L}, \alpha \rangle$. Take $\mathcal{L} \in C^+_X$ such that $\langle \mathcal{L}, \alpha \rangle < 0$. Then by HAG, Corollary V.1.8, $\mathcal{L}^n$ is effective for some $n > 0$. Since $\alpha$ is nef, $n(\alpha, \mathcal{L}) = \langle \alpha, \mathcal{L}^n \rangle \geq 0$, which is a contradiction. \qed

**Proposition 5.6.** Let $X$ be a smooth projective surface over a field $k$. Then

$$\text{Amp}(X) = \text{Int Nef}(X) \subset \text{Nef}(X) = \overline{\text{Amp}(X)}$$

where “Int” stands for “interior”.

**Proof.** The nef cone is closed by definition, so it is enough to show that $\text{Amp}(X) = \text{Int Nef}(X)$. The ample cone is open. Indeed, if $H$ is an ample class, then $\langle H, H \rangle \geq 1$ and $\langle H, C \rangle \geq 1$ for any curve $C$. Obviously $\text{Amp}(X) \subset \text{Nef}(X)$, therefore $\text{Amp}(X) \subset \text{Int Nef}(X)$. On the other hand, if $\alpha \in \text{Int Nef}(X)$, then $\alpha - \epsilon H$ is still in the nef cone for some $\epsilon > 0$ not too big. Then $\alpha = \alpha - \epsilon H + \epsilon H$ is a sum of a nef and an ample class, which is easily checked to be ample (a consequence of Proposition 5.5 and Remark 5.4, iii)) \qed

We consider now the case where $X$ is a $K_3$ surface over a field $k$. Then the adjunction formula reads

$$(C.C) = 2g(C) - 2$$

for all curves $C \subset X$. Therefore either $(C.C) = -2$, which happens when $C \cong \mathbb{P}_k^1$, or $(C.C) \geq 0$.

The following is a corollary of Nakai-Moishezn-Kleiman Theorem.
Corollary 5.7. A line bundle $\mathcal{L}$ on a $K_3$ surface $X$ over a field $k$ is ample if and only if

i) $(\mathcal{L})^2 > 0$,

ii) $(\mathcal{L}.C) > 0$ for every smooth rational curve $\mathbb{P}^1 \cong C \subset X$, and

iii) $(\mathcal{L}.\mathcal{H}) > 0$ for one ample line bundle $\mathcal{H}$ (or, equivalently, for all of them).

Proof. If $\mathcal{L}$ is ample, then i), ii), iii) are satisfied. On the other hand, for every non-rational curve on $X$, $(C.C) \geq 0$. Moreover, $(\mathcal{H}.C) > 0$ for any $\mathcal{H}$ ample, hence $C \in \mathcal{C}_X$. By i) and iii) it also follows that $\mathcal{L} \in \mathcal{C}_X$. Now, by Remark 5.2 iii) it follows that $(\mathcal{L}.C) > 0$. Therefore $(\mathcal{L}.C) > 0$ for all curves $C \subset X$, and we conclude by Nakai-Moishezon-Kleiman Theorem.

Corollary 5.8. For a $K_3$ surface over a field $k$ one has

$$\text{Amp}(X) = \{ \alpha \in \mathcal{C}_X^+ : (\alpha.C) > 0 \text{ for all } \mathbb{P}^1 \cong C \subset X \}.$$  

6 Chambers and walls

Let $X$ be a $K_3$ surface over a field $k$. We have seen that for a curve $C \subset X$, $(C.C) \geq -2$ and $(C.C) = -2$ if and only if $C$ is smooth and rational. We define the subset of $-2$-classes of the Neron-Severi group $\text{NS}(X)$ (which, for a $K_3$ surface is isomorphic to $\text{Num}(X)$ and $\text{Pic}(X)$):

$$\Delta := \{ \delta \in \text{NS}(X) : \delta^2 = -2 \}$$

Let $\mathcal{L}$ be the class of a line bundle in $\Delta$. By Riemann-Roch theorem we have

$$\chi(\mathcal{L}) = h^0(\mathcal{L}) - h^1(\mathcal{L}) + h^2(\mathcal{L}) = \chi(\mathcal{O}_X) + (\mathcal{L})^2/2 + (\mathcal{L}.\omega_{X/k}^{-1})/2 = 2 + (\mathcal{L})^2/2.$$  

Serre duality and triviality of $\omega_{X/k}$ imply that

$$h^0(\mathcal{L}) + h^0(\mathcal{L}^{-1}) \geq 2 + (\mathcal{L})^2/2 = 1.$$  

Hence exactly one of $\mathcal{L}$ and its dual is effective. Notice also that, since for a $K3$-surface $\text{NS}(X) = \text{Pic}(X)$, effective divisor classes are well defined in $\text{NS}(X)$.

Then we let $\Delta = \Delta^+ \cup \Delta^-$ where

$$\Delta^+ := \{ \delta \in \text{NS}(X) : \delta^2 = -2 \text{ and } \delta \text{ is an effective divisor class} \}, \quad \Delta^- := -\Delta^+.$$  

We let $\delta^\perp \subset \text{NS}(X)_k$ be the hyperplane orthogonal to $\delta \in \Delta^+$.

Definition 6.1. The connected components of

$$\mathcal{C}_X^+ \setminus \bigcup_{\delta \in \Delta^+} \delta^\perp$$

are called the chambers of $\mathcal{C}_X^+$, while the $\delta^\perp$ are the walls of the chambers.

Remark 6.2. The union of the walls is closed in $\text{NS}(X)_\mathbb{R}$, and the collection of the walls is locally finite in $\text{NS}(X)_\mathbb{R}$. [2, Chapt. 8, Remark 2.2]

Proposition 6.3. The ample cone $\text{Amp}(X)$ is a chamber of $\mathcal{C}_X^+$. 

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Proof. Let
\[ \aleph := \{ \alpha \in C_X^+ : (\alpha, \delta) > 0 \text{ for all } \delta \in \Delta^+ \}. \]
This is a chamber of \( C_X^+ \). Classes \( \alpha \) in \( \text{Amp}(X) \) are given by the conditions \( \alpha \in C_X^+ \) and \( (\alpha, C) > 0 \) for all curves in \( X \) with \( C.C = -2 \). Hence \( \aleph \subset \text{Amp}(X) \). On the other hand, if \( H \in \text{Amp}(X) \), then \( (H, \delta) > 0 \) for all \( \delta \in \Delta^+ \) since \( \delta \) is an effective divisor class. So \( \text{Amp}(X) = \aleph \), which is a chamber of \( C_X^+ \).

Now, for all \( \delta \in \Delta \) consider the isometry on \( \text{NS}(X)_{\mathbb{R}} \) given by
\[ s_\delta(x) = x + (x, \delta)\delta. \]
Then \( s_\delta(\delta) = -\delta \) and \( s_\delta = \text{id} \) on \( \delta^\perp \). If \( x \in C_X^+ \), then \( s_\delta(x)^2 > 0 \). Moreover \( (s_\delta(x), x) > 0 \), hence \( s_\delta(x) \in C_X^+ \). Therefore the \( s_\delta \)'s preserve the positive cone.

Definition 6.4. The subgroup \( W \) of the orthogonal group \( O(\text{NS}(X)_{\mathbb{R}}) \) generated by \( \{ s_\delta : \delta \in \Delta \} \) is called Weyl group.

The Weyl group \( W \) preserves the union of the walls \( \bigcup_{\delta \in \Delta^+} \delta^\perp \). Indeed one can check that \( W(\Delta) = \Delta \), and if \( x \in \delta^\perp \), then \( s_{x'}(x) \in s_{x'}(\delta)^\perp \) for all \( x' \in \Delta \). Therefore \( W \) acts on the set of chambers of \( C_X^+ \).

Proposition 6.5. The Weyl group acts simply transitively on the set of chambers of \( C_X^+ \). The cone \( \text{Nef}(X) \cap C_X^+(X) \) is a fundamental domain for the action of the Weyl group on \( C_X^+(X) \).

Proof on [2, Chapt. 8, Prop. 2.6. and Corollary 2.11]

References

[1] Robin Hartshorne, \textit{Algebraic Geometry}  
Berlin, New York: Springer-Verlag, 1977

[2] Daniel Huybrechts \textit{Lectures on K3 surfaces}