Further results on ample and nef cones – Erik Visse

These are the notes from the seminar on K3 surfaces and their automorphisms held in Leiden at the end of 2014. The website for this seminar can be found at `http://pub.math.leidenuniv.nl/~vissehd/k3seminar/`.

Recall that for a surface X, we write $\mathrm{Num}(X)$ for $\mathrm{Pic}(X)/\equiv$, where $\equiv$ denotes numerical equivalence. We write $\mathrm{Num}(X)_{\mathbb{R}}$ for the vector space $\mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

Warning 0.1. A lot of the results in this talk come from [Laz04] which uses the notation $N^1(X)$ for $\mathrm{Num}(X)$ and calls it the Néron–Severi group of X. Also, Lazarsfeld treats intersection theory in a greater generality, namely also for higher dimensional schemes, but this should not pose a problem for the reader.

Remark 0.2. Whenever we say 'divisor', we mean 'Cartier divisor'.

1 Ampleness

In a Orecchia's lecture we have seen the Nakai–Moishezon–Kleiman criterion for ampleness of a divisor D, we repeat it here for convenience.

Theorem 1.1 (NMK). *Let X be a projective surface and D a divisor on X. Then D is ample if and only if both $D^2 > 0$ and $D \cdot C > 0$ for all curves C on X.*

This criterion shows that ampleness is a numerical property and we can talk about ampleness of classes in $\mathrm{Num}(X)$. We start this talk with some other criteria for ampleness.

Proposition 1.2 (Laz04, Ex. 1.2.32). *Let X be a projective surface and D a divisor. Then D is ample if and only if one of the following holds:*

- *for every subvariety V of X of positive dimension, there is an integer $m = m(V)$ and a non-zero section $s_V \in H^0(V, \mathcal{O}_V(mD))$ such that s_V vanishes at some point of V; or*
- *for every subvariety V of X of positive dimension, $\chi(V, \mathcal{O}_V(mD)) \to \infty$ as $m \to \infty$.*

It is non-trivial, but true, that the Nakai–Moishezon–Kleiman criterion extends to $\mathrm{Num}(X)_{\mathbb{R}}$.

Proposition 1.3 (Laz04, Cor. 1.4.11). *Let X be a projective surface and H a given $\mathbb{R}$-divisor on X. Then an $\mathbb{R}$-divisor D on X is ample if and only if there exists a positive $\varepsilon > 0$ such that*

$$\frac{D \cdot C}{H \cdot C} \geq \varepsilon$$

holds for all curves C on X.

For a K3 surface, we can leave out the requirement $D^2 > 0$ from the Nakai–Moishezon–Kleiman criterion for ampleness (i.e. it is automatic). To prove this over the complex numbers will be the topic of the next part.

1.1 NMK for K3s

We begin this section with quite a powerful theorem. It may be possible to achieve our goal in a simpler way, but I am not aware of it.

Theorem 1.4 (Fujita's vanishing theorem, Laz04, Thm. 1.4.35). *Let X be a complex projective scheme and let H be an (integral) ample divisor on X. Given any coherent sheaf $\mathcal{F}$ on X, there exists an integer $m(\mathcal{F}, H)$ such that*

$$H^i(X, \mathcal{F}(mH + D)) = 0$$

holds for all $i > 0$, all $m \geq m(\mathcal{F}, H)$ and any nef divisor D on X.

From this theorem, we find an asymptotic growth formula for some cohomology groups.

Theorem 1.5 (Laz04, Thm. 1.4.40). *Let X be a complex projective scheme of dimension n and D a nef divisor on X. Then for any coherent sheaf $\mathcal{F}$ on X, we have*

$$h^i(X, \mathcal{F}(mD)) = O(m^{n-i}).$$

Proof. Here we only give a sketch of the proof as it goes by induction to the dimension n and we have not treated (nor do we plan to treat) intersection theory on schemes other than surfaces. Also, since the proof from Lazarsfeld is somewhat sketchy, a lot of details are still left out.

Since for each $m \in \mathbb{Z}_{>0}$, mD is nef, by Fujita's vanishing theorem there exists some ample divisor H such that $h^i(X, \mathcal{F}(H + mD)) = 0$ holds for $i > 0$ and all m (take any ample H' and $H = nH'$ where $n \geq m(\mathcal{F}, H')$ and further take n such that H is effective). Assuming that the support of H does not contain any of the subvarieties defined by associated primes of $\mathcal{F}$, we have an exact sequence

$$0 \to \mathcal{F}(mD) \to \mathcal{F}(mD + H) \to \mathcal{F}(mD + H) \otimes \mathcal{O}_H \to 0 \tag{1}$$

by tensoring the sequence

$$0 \to \mathcal{O}(-H) \to \mathcal{O}_X \to \mathcal{O}_H \to 0$$

with $\mathcal{F}(mD + H)$.

[After the talk, Maxim Mornev supplied me with a sketch of why the sequence in 1 is exact, but any mistakes are still my own. We need to check injectivity of the first non-trivial map. Since H is Cartier, it is locally given by some equation f. Furthermore, locally $\mathcal{F}$ is given by a module M and the map $\mathcal{F}(mD) \to \mathcal{F}(mD+H)$ is induced by multiplication by f on M. Since H does not contain any of the subvarieties defined by associated primes of $\mathcal{F}$, multiplication by f is injective by [Stacks Tag 00LD].]

From 1 and Theorem 1.4, we find

$$h^i(X, \mathcal{F}(mD)) \leq h^{i-1}(H, \mathcal{F}(mD + H) \otimes \mathcal{O}_H))$$

and we can conclude the proof by induction to the dimension and [Laz04, Ex.1.2.33] for the case $i - 1 = 0$. ☐

Now we have enough to prove

Theorem 1.6. *Let X be a K3 surface over $\mathbb{C}$ and H a divisor on X. Then H is ample if and only if $H \cdot C > 0$ holds for all curves C on X.*

Proof. We adapt the proof that Hartshorne gives for the Nakai–Moishezon–Kleiman criterion [Har77, Thm. V.1.10]. Here, the requirement $H^2 > 0$ is only needed to asure that nH will be effective for large enough n. If we show that via some other method, we can finish the proof as Hartshorne does.

The Riemann–Roch theorem for surfaces reads

$$h^0(D) - h^1(D) + h^2(D) = \frac{1}{2} D \cdot (D - K) + \chi(0),$$

which in the case for K3 surfaces (using Serre duality) reduces to

$$h^0(D) + h^0(-D) = \frac{1}{2} D^2 + 2 + h^1(D).$$

By assumption, H is nef, so by Theorem 1.5, we have $h^1(nH) = O(n)$ and we find

$$h^0(nH) + h^0(-nH) = 2 + \frac{1}{2} n^2 H^2 + O(n) \tag{2}$$

Since the two terms on the left-hand side of equation (2) are non-negative, so is the right-hand side. We distinguish two cases: 1) $H^2 < 0$ and 2) $H^2 = 0$.

1. In this case we would get $h^0(nH) + h^0(-nH) \to -\infty$ for $n \to \infty$ which contradicts that this is always non-negative.
2. In this case we have $h^0(nH) + h^0(-nH) > 0$ for large enough n. Take such an n and let C be any curve on X not contained in the support of H. Then by assumption, $H \cdot C > 0$ holds and $(-nH) \cdot C = -n(H \cdot C) < 0$. As C is not contained in the support of H (which is equal to the support of $-nH$), we conclude that $-nH$ cannot be effective and therefore that nH is.

□

2 Nefness

We continue with some results about nef divisors and cones.

Definition 2.1. A smooth projective surface X is called minimal if X does not contain any (-1)-curves, i.e. rational smooth curves of self-intersection -1.

Proposition 2.2 (Laz04, Ex.1.4.18). *Let X be a smooth projective surface of non-negative Kodaira dimension. Then X is minimal if and only if K_X is nef.*

Proof. We will make use of the adjunction formula, which for possibly non-smooth curves C reads

$$C^2 + C \cdot K_X = 2p_a(C) - 2$$

where $p_a(C) = 1 - \chi(\mathcal{O}_C)$ is its arithmetic genus.

Assume that X is minimal and suppose that we have a curve C such that $K_X \cdot C < 0$ holds. Take any $D \in |mK_X| \neq \emptyset$ and write $D = \sum a_i C_i$ for $a_i > 0$ and C_i irreducible. Then C must be one of the C_i, say $C = C_1$ and we have

$$a_1(C \cdot C) \leq D \cdot C = K_X \cdot C < 0$$

from which we conclude $C^2 < 0$. From the adjunction formula we then find $C^2 = -1$ and $p_a(C) = 0$. The latter can only occur of C is smooth and rational. This contradicts our assumption.

Conversely assume that K_X is nef and suppose that C is a (-1)-curve. Then the adjunction formula gives

$$C \cdot K = 2p_a(C) - 2 - C^2 = -1$$

which contradicts that K_X is nef. $\square$

Definition 2.3. Let X be a complete surface. The cone of curves of X is

$$\mathrm{NE}(X) = \left\{ \sum a_i [C_i] \middle| \forall i : C_i \text{ is an irreducible curve}, a_i > 0 \right\} \subset \mathrm{Num}(X)_{\mathbb{R}}$$

and its closure $\overline{\mathrm{NE}}(X)$ is called the closed cone of curves.

Definition 2.4. Let V be a finite dimensional real vector space and K a closed convex cone in V. The dual cone of K is

$$K^\vee = \left\{ \phi \in V^\vee \middle| \phi(x) \geq 0 \forall x \in K \right\}.$$

It is a closed cone in $V^\vee$.

Fact 2.5. Under the natural identification $V^{\vee\vee} = V$, one has $K^{\vee\vee} = K$.

Proposition 2.6. *Let X be a complete surface. Then $\overline{NE}(X) = \mathrm{Nef}(X)^\vee$, i.e.*

$$\overline{\mathrm{NE}}(X) = \{ D \in \mathrm{Num}(X)_{\mathbb{R}} | \, D \cdot E \geq 0 \ \forall E \in \mathrm{Nef}(X) \}.$$

Proof. From the definition of the dual cone and Fact 2.5. $\square$

2.1 Examples of $\mathrm{Nef}(X)$ and $\overline{\mathrm{NE}}(X)$

Example 2.7 (Laz04, Ex.1.4.7). Let X be a complete surface, and suppose that a connected algebraic group G acts transitively on X. Then any effective divisor D is nef. To see this, let C be a curve on X. For a general element $g \in G$, the translate gD does intersects C transversally. Since G is connected, we have $gD \equiv D$ and therefore

$$D \cdot C = (gD) \cdot C \geq 0.$$

Let C be a projective curve over a field k, and let $X = C \times C$. Let $P \in C(k)$. Then there are at three obvious classes in $\mathrm{Num}(X)_{\mathbb{R}}$

$$f_1 = [\{P\} \times C], \; f_2 = [C \times \{P\}], \; \delta = [\Delta]$$

where $\Delta \subset C \times C$ is the diagonal. Their intersection pairings are as follows:

$$\begin{aligned} \delta \cdot f_1 = \delta \cdot f_2 = f_1 \cdot f_2 &= 1, \\ f_1^2 = f_2^2 &= 0, \\ \delta^2 &= 2 - 2g, \end{aligned}$$

where $g = g(C)$ is the genus of C. It follows that f_1, f_2, δ are independent elements of $\mathrm{Num}(X)_{\mathbb{R}}$

FACT 2.8. If C is sufficently general, then f_1, f_2, δ span $\mathrm{Num}(X)_{\mathbb{R}}$.

EXAMPLE 2.9. In the case $g(C) = 1$, X is an abelian surface and we have

PROPOSITION 2.10 (Laz04, Lem.1.5.4).

$$\overline{\mathrm{NE}}(X) = \mathrm{Nef}(X)$$

Furthermore, a class α is nef if and only if $\alpha^2 \geq 0$ and $\alpha \cdot h \geq 0$ hold for some ample class h. In particular, writing $\alpha = xf_1 + yf_2 + z\delta$, then α is nef if and only if

$$\begin{aligned} xy + yz + zx &\geq 0 \text{ and} \\ x + y + z &\geq 0 \end{aligned}$$

are both satisfied.

Proof. The first part is a special case of Example 2.7. Using this, the 'only if'-part of the second statement follows. Lazarsfeld claims that the 'if'-part follows from [Har77, Cor.V.1.8] which says that for a(n integral) divisor D such that $D^2 > 0$ and $D \cdot H > 0$ hold for some ample H, that nD is linearly equivalent to an effective divisor for suitably large n. I have so far not yet been able to consider the case where either of the inequalities in the statement of the proposition is equality, but I guess there is an approximation argument involved.

The given equations follow by taking $h = f_1 + f_2 + \delta$ for which ampleness can be checked by working out all the necessary intersection products. □

Let X be a K3 surface. We fix some notation.

DEFINITION 2.11. $\mathcal{N}(X)$ is the set of nodal classes, i.e. the classes of smooth rational curves (those have self-intersection -2). $\mathcal{E}(X)$ is the set of rational curves of self-intersection 0.
Further, for an ample class h, we denote

$$\mathcal{Q}(X) = \left\{ D \in \mathrm{Num}(X)_{\mathbb{R}} \,|\, D^2 = 0, D \cdot h \geq 0 \right\}.$$

LEMMA 2.12. *$\mathcal{Q}(X)$ does not depend on the choice of h.*

Proof. This follows from Proposition 1.3. □

THEOREM 2.13 (Kov94, Thm.2). *Let X be a K3 surface. Then one of the following holds:*

- *$\rho(X) = 1$ and $\overline{\mathrm{NE}}(X) = \mathbb{R}_+ h$ where h is an ample class,*

- *$\rho(X) = 2$ and $\overline{\mathrm{NE}}(X) = \mathbb{R}_+[e] + \mathbb{R}_+[l]$ where $e \in \mathcal{E}(X)$ and $l \in \mathcal{N}(X)$,*
- *$2 \leq \rho(X) \leq 4$ and $\overline{\mathrm{NE}}(X) = Conv(\mathcal{Q}(X))$ and $\partial\overline{\mathrm{NE}}(X)$ does not contain any effective class, i.e. X contains neither smooth rational nor smooth genus 1 curves,*
- *$2 \leq \rho(X) \leq 11$ and $\overline{\mathrm{NE}}(X) = Conv(\mathcal{Q}(X)) = \overline{\sum_{e\in\mathcal{E}(X)} \mathbb{R}_+[e]}$, in particular X does not contain a smooth rational curve,*
- *$2 \leq \rho(X) \leq 20$ and $\overline{\mathrm{NE}}(X) = \overline{\sum_{l\in\mathcal{N}(X)} \mathbb{R}_+[l]}$*

and all of these possibilities occur.

From this, one can immediately conclude the following theorem.

Theorem 2.14. *Let X be a K3 surface with Picard number at least 3. Then one of the following is satisfied:*

- *X does not contain any curve of negative self-intersection, or*
- $\overline{\mathrm{NE}}(X) = \overline{\sum_{l\in\mathcal{N}(X)} \mathbb{R}_+[l]}$.

References

[Bad01] Lucian Bădescu, *Algebraic Surfaces*, Springer-Verlag, Berlin Heidelberg, 2001

[Bea96] Arnaud Beauville, *Complex Algebraic Surfaces*, Cambridge University Press, 1996

[BHPV04] Wolf P. Barth, Klaus Hulek, Chris A.M. Peters, Antonius van de Ven, *Compact Complex Surfaces*, Springer-Verlag, Berlin Heidelberg, 2004

[Har77] Robin Hartshorne, *Algebraic Geometry*, Springer-Verlag, Berlin Heidelberg, 1977

[Laz04] Robert Lazarsfeld, *Positivity in Algebraic Geometry I*, Springer-Verlag, Berlin Heidelberg, 2004

[Kov94] Sándor J. Kovács, The cone of curves of a K3 surface, *Mathematische Annalen* 300, p.681-691, 1994

[Stacks] The Stacks Project authors, *The Stacks Project*, http://stacks.math.columbia.edu/