Finiteness results on automorphism groups of K3 surfaces – Erik Visse

These are the notes from the seminar on K3 surfaces and their automorphisms held in Leiden at the end of 2014. The website for this seminar can be found at `http://pub.math.leidenuniv.nl/~vissehd/k3seminar/`.

As always, we will work over the complex numbers and K3 surfaces are assumed to be projective. Everything I will do today can (in more or less detail) be found in [Huy14].

1 Symplectic automorphisms

We begin by recalling the Torelli theorem for K3 surfaces, where we consider $X = X'$:

Theorem 1.1. *(Global Torelli for a K3 surface) Let X be a K3 surface. There is an isomorphism*

$$\mathrm{Aut}(X) \cong \{\textit{Hodge isometries of } H^2(X,\mathbb{Z}) \textit{ preserving the ample cone}\}.$$

In this talk, we will also heavily use the Hodge structure on the transcendental lattice $\mathrm{T}(X)$. Recall that we have defined the transcendental lattice as the part of $H^2(X,\mathbb{Z})$ orthogonal to the Néron–Severi group $\mathrm{NS}(X) = H^(X,\mathbb{Z}) \cap H^{1,1}(X)$. There exists an alternative, equivalent definition.

Definition 1.2. The transcendental lattice is the minimal primitive sub-Hodge structure of $H^2(X,\mathbb{Z})$ such that $H^{2,0}(X) = T^{2,0} \subset T_{\mathbb{C}}$ holds.

We first consider a subgroup of the full automorphism group of a K3 surface.

Definition 1.3. Let X be a K3 surface. An automorphism $f : X \to X$ is called *symplectic* if the induced $f^* : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ acts on $H^0(X,\Omega_X^2) = H^{2,0}(X)$ as the identity. We denote the group of symplectic automorphisms of X as $\mathrm{Aut}_s(X)$.

Remark 1.4. This definition is equivalent to saying that f^* acts on $\mathrm{T}(X)$ as the identity. From the inclusion $H^{2,0}(X) \subset \mathrm{T}(X)_{\mathbb{C}}$, one direction of this equivalence is obvious. For the other direction, one uses that for symplectic f, $\ker((f^*-\mathrm{id})|_{\mathrm{T}(X)}) \subset \mathrm{T}(X)$ is a primitive sub-Hodge structure containing $H^{2,0}$ combined with the alternative definition of $\mathrm{T}(X)$.

To be able to proof some of the facts, we need the following lemma, which we state without proof.

Lemma 1.5 (Huy14, Lem 15.1.4). *Let X be a K3 surface and let $f \in \mathrm{Aut}_s(X)$ have finite order n. Suppose that $x \in X$ is a fixed point of f. Then there exist local holomorphic coordinates z_1, z_2 around x such that $f(z_1,z_2) = (\lambda_x z_1, \lambda_x^{-1} z_2)$ holds for some primitive nth root of unity λ_x.*

Corollary 1.6 (Huy14, Cor 15.1.5). *Let $f \in \mathrm{Aut}_s(X)$ be of finite order $n > 1$. Denote the set of fixed points of f by $\mathrm{Fix}(f)$. Then $1 \le |\,\mathrm{Fix}(f)| \le 8$ holds and more* *precisely*

$$|\operatorname{Fix}(f)| = \frac{24}{n}\prod_{p|n}\left(1+\frac{1}{p}\right)^{-1}.$$

Proof. By the above lemma, the fixed points of f are isolated, and we may apply the holomorphic Lefschetz fixed-point formula and get

$$\begin{aligned}\sum_i (-1)^i \operatorname{tr}(f^*|_{H^i(X,\mathcal{O}_X)}) &= \sum_{x\in\operatorname{Fix}(f)} \det(\operatorname{id} - d_x f)^{-1} \\ &= \sum_{x\in\operatorname{Fix}(f)} \frac{1}{(1-\lambda_x)(1-\lambda_x^{-1})}.\end{aligned}$$

Since $H^i(X,\mathcal{O}_X)$ is one-dimensional for $i=0,2$ and zero otherwise and since f is symplectic, the left hand side equals 2 and in particular the set of fixed-points in non-empty. Since $|\lambda_x| \le 1$ holds, we have $|1-\lambda_x^{\pm 1}| \le 2$ and therefore $|\operatorname{Fix}(f)| \le 8$.

If k is prime to n, then $\operatorname{Fix}(f^k) = \operatorname{Fix}(f)$ holds and hence we get

$$\sum_{x\in\operatorname{Fix}(f)} \frac{1}{\varphi(n)} \sum_{\gcd(k,n)=1} \frac{1}{(1-\lambda_x^k)(1-\lambda_x^{-k})} = 2.$$

By a useful formula from number theory, we have

$$\sum_{\gcd(k,n)=1} \frac{1}{(1-\lambda_x^k)(1-\lambda_x^{-k})} = \frac{n^2}{12}\prod_{p|n}\left(1-\frac{1}{p^2}\right)$$

which combined with the well-known expression $\varphi(n) = n\prod_{p|n}\left(1-\frac{1}{p}\right)$ yields the result. □

Corollary 1.7 (Huy14, Cor 15.1.8). *For an symplectic automorphism f of order $n>1$, the only possibilities that may occur are given in the following table.*

$n =$	*2*	*3*	*4*	*5*	*6*	*7*	*8*
$\lvert\operatorname{Fix}(f)\rvert =$	*8*	*6*	*4*	*4*	*2*	*3*	*2*
$\rho(X) \ge$	*9*	*13*	*15*	*17*	*17*	*19*	*19*

Proof. This essentially follows from Corollary **??** combined with finding a lower bound $\dim H^\bullet(X,\mathbb{C})^{\langle f\rangle} \ge 5$ by collecting linear independent subspaces. These are tied together by the topological Lefschetz trace formula. For slightly more details, see [Huy14, Cor 15.1.8 and further]. □

2 Hodge isometries of $\mathrm{T}(X)$

Next, we will describe the Hodge isometries of $\mathrm{T}(X)$ in order to compare $\operatorname{Aut}_s(X)$ and $\operatorname{Aut}(X)$.

Lemma 2.1 (Huy14, Lem 3.4.3). *If $\alpha : \mathrm{T}(X) \to \mathrm{T}(X)$ is a Hodge isometry such that $\alpha|_{T^{2,0}} = \mathrm{id}_{T^{2,0}}$, then $\alpha = \mathrm{id}$.*

Proof. Essentially follows immediately from the minimality of $\mathrm{T}(X)$. □

Proposition 2.2. *Let $\alpha : \mathrm{T}(X) \to \mathrm{T}(X)$ be a Hodge isometry. Then there exists an integer $n \geq 1$ such that $\alpha^n = \mathrm{id}$. Furthermore, the group of Hodge isometries of $\mathrm{T}(X)$ is finite and cyclic.*

Proof. This is [Huy14, Cor 3.4.4]. □

Corollary 2.3 (Huy14, Cor 15.1.10). *Let X be a K3 surface and $f \in \mathrm{Aut}(X)$. Then there exists an integer $n \geq 1$ such that f^{*n} is the identity on $\mathrm{T}(X)$.*

Proof. Immediate from Proposition **??**. □

Proposition 2.4 (Huy14, Rmk 15.1.12). *The inclusion $\mathrm{Aut}_s(X)$ into $\mathrm{Aut}(X)$ extends to an exact sequence*

$$1 \to \mathrm{Aut}_s(X) \to \mathrm{Aut}(X) \to \mu_m \to 1.$$

Proof. This is the second statement in Proposition **??** which translates as "the cokernel of the inclusion $\mathrm{Aut}_s(X) \to \mathrm{Aut}(X)$ is finite and cyclic". □

Consider the natural map $\phi : \mathrm{Aut}(X) \to O(\mathrm{NS}(X))$ sending f to $f^*|_{\mathrm{NS}(X)}$, where $f^* \in O(H^2(X, \mathbb{Z}))$.

Proposition 2.5. *The kernel of ϕ is isomorphic to a subgroup μ_n of μ_m.*

Proof. Denote the map $\mathrm{Aut}(X) \to \mu_m$ by g and the map $g \circ (\ker \phi \hookrightarrow \mathrm{Aut}(X))$ by ψ. For any $x \in \ker \phi$ with $\psi(x) = 1$, we have $g(x) = 1$ and therefore $x \in \mathrm{Aut}_s(X)$. For such x, we have $x^*|_{\mathrm{T(X)}} = \mathrm{id}\,|_{T(X)}$ from $x \in \mathrm{Aut}_s(X)$ and $x^*|_{\mathrm{NS}(X)} = \mathrm{id}_{\mathrm{NS}(X)}$ from $x \in \ker \phi$. Since $\mathrm{NS}(X) \oplus \mathrm{T}(X)$ is of finite index in $H^2(X, \mathbb{Z})$ and the latter is torsion-free, we have $x^* = \mathrm{id} \in O(H^2(X, \mathbb{Z}))$ and by the Torelli theorem, $x = \mathrm{id} \in \mathrm{Aut}(X)$. Therefore the map ψ is injective and $\ker \phi$ can be identified with a subgroup of μ_m, which necessarily is μ_n for some $n \leq m$. □

If $n = m$ holds, then $\psi : \ker \phi \to \mu_m$ is an isomorphism, and the sequence in Proposition **??** splits. Conversely, if it splits then we have $n = m$.

Proposition 2.6 (Huy14, Cor 15.1.14). *The order of the cyclic group μ_m, which is m, satisfies*

$$\varphi(m) \leq \mathrm{rk}\, \mathrm{T}(X) = 22 - \rho(X)$$

and moreover $\varphi(m) | \mathrm{rk}\, \mathrm{T}(X)$. Furthermore, we have $m \leq 66$.

Proof. Let $f \in \mathrm{Aut}(X)$ act on $\mathrm{T}(X)$ by a primitive mth root of unity ζ_m. Then the minimal polynomial of ζ_m, which is Φ_m, divides the characteristic polynomial of f^* on $\mathrm{T}(X)$. Hence $\varphi(m) = \deg \Phi_m \leq \mathrm{rk}\, \mathrm{T}(X)$ holds. To prove $\varphi(m) | \mathrm{rk}\, \mathrm{T}(X)$, we need some representation theory. This proof still lacks some details. They will be filled in later. We consider the irreducible sub-representations of μ_m on $\mathrm{T}(X) \otimes \mathbb{Q}$.

Suppose that one of them has rank different from $\varphi(m)$. Then there exists an $n < m$ and an $\alpha \in \mathrm{T}(X)$ with $f^{n*}(\alpha) = \alpha$. For every $\beta \in H^{2,0}(X)$, we have $(\alpha, \beta) = (f^{n*}(\alpha), f^{n*}(\beta)) = (\alpha, f^{n*}(\beta))$. So by non-degeneracy of the intersection pairing on $H^2(X, \mathbb{Z})$, we have $\alpha \in H^{1,1}(X)$, which is a contradiction. Thus as a representation of μ_m one has $\mathrm{T}(X) \cong \mathbb{Z}[\zeta_m]^r$ where necessarily $r = \operatorname{rk} \mathrm{T}(X)/\varphi(m)$. The bound $m \leq 66$ follows since for $m > 66$ we have $\varphi(m) > 21$. □

3 Possible finiteness of $\mathrm{Aut}(X)$

Proposition 3.1. *Let X be a K3 surface. Then $\mathrm{Aut}(X)$ is finitely generated.*

Proof. This is [Huy14, Cor 15.2.4]. □

For the next result, we need to recall the Weyl group, which is a subgroup of $O(\mathrm{NS}(X))$. Recall that for any (-2)-class δ in $\mathrm{NS}(X)$, we have defined the reflection through δ as the isometry $R_\delta(D) = D + (\delta \cdot D)\delta$. Let Δ be the set of (-2)-classes. Then the Weyl group is $W = \langle R_\delta : \delta \in \Delta \rangle \subset O(\mathrm{NS}(X))$. This definition also makes sense for any even lattice.

Remark 3.2. We have seen (from Riemann-Roch for surfaces) that for $\delta \in \Delta$ either δ or $-\delta$ is effective. It may however still be that neither is the class of an actual (-2)-curve, i.e. a smooth rational curve.

The following theorem is due to Piatetsky-Shapiro and Shafarevich. Unfortunatly, I could not find their paper (in a language that I read and understand) and Huybrechts only sketches the proof.

Theorem 3.3 (Huy14, Thm 15.2.6). *Let X be a K3 surface. The natural map $f \mapsto f^*$ induces a homomorphism $\mathrm{Aut}(X) \to O(\mathrm{NS}(X))/W$ with finite kernel and cokernel. In particular, the former is finite if and only if the latter is finite.*

Definition 3.4. Let $\mathcal{C}^\rho$ denote the set of isomorphism classes of even lattices N of signature $(1, \rho - 1)$ with finite $O(N)/W$.

The following is a result by Nikulin.

Theorem 3.5 (Huy14, Thm 15.2.10). *The set $\mathcal{C}^\rho$ is empty for $\rho \geq 20$ and non-empty and finite for $2 \leq \rho \leq 19$. Every $N \in \mathcal{C}^\rho$ can be realized as the Néron–Severi lattice of some K3 surface.*

The above theorem states that any K3 surface with $\rho = 20$ has an infinite automorphism group. It also states that for $2 \leq \rho \leq 19$, there are only finitely many possibilities for $\mathrm{Aut}(X)$ given that it is finite. A quite explicit list of possibilities for N (at least for $\rho \geq 5$) is found in [Nik14].

For $\rho = 1$ there is even something better:

Proposition 3.6 (Huy14, Cor 15.2.12). *Let X be a K3 surface with $\mathrm{Pic}(X) \cong \mathbb{Z} \cdot H$.*

- *If $H^2 > 2$, then $\mathrm{Aut}(X) = \{\mathrm{id}\}$,*

- *if* $H^2 = 2$*, then* $\mathrm{Aut}(X) \cong \mathbb{Z}/2\mathbb{Z}$*.*

Proof. This is [Huy14, 15.2.12]. □

There exits a complete classification of possible finite subgroups of $\mathrm{Aut}_s(X)$. We treat it in two parts.

THEOREM 3.7 (Huy14, Thm 15.2.13). *Suppose* $\mathrm{Aut}(X)$ *is finite. Then* $\mathrm{Aut}_s(X)$ *is isomorphic to one of the following groups:* $\{1\}$, $\mathbb{Z}/2\mathbb{Z}$*, or* S_3*.*

If we allow $\mathrm{Aut}(X)$ to be infinite, then the classification becomes more difficult and is due to Mukai.

THEOREM 3.8 (Huy14, Thm 15.3.1). *For a finite group G the following conditions are equivalent:*

1. *There exists a K3 surface X such that G is isomorphic to a subgroup of* $\mathrm{Aut}_s(X)$*.*
2. *There exists an injection $G \hookrightarrow M_{23}$ into the Mathieu group M_{23} such that the induced action of G on $\{1, \dots, 24\}$ has at least five orbits.*

This last part probably needs some explanation. The Mathieu group M_{23} is a subgroup of the Mathieu group M_{24} which acts 5-transitively on 24 elements. M_{23} is the stabilizer in M_{24} of an element and therefore acts 4-transitively on a set of 23 elements. M_{23} has $2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23 = 10.200.960$ elements and is one of the sporadic simple groups (as is M_{24}).

The connection between K3 surfaces and the Mathieu groups M_{24} and M_{23} (called 'Mathieu moonshine') is largely unexplained and seems to be of large interest to mathematical physicists.

That there is such a large difference between the two cases above (finite $\mathrm{Aut}(X)$ or not) is a sign that K3 surfaces with finite automorphism groups are rather special. If we only consider the possible finite abelian subgroups of $\mathrm{Aut}_s(X)$, the picture becomes a lot clearer again.

THEOREM 3.9 (Huy14, Thm 15.3.14). *There are exactly 15 finite abelian groups G that can occur as subgroups of* $\mathrm{Aut}_s(X)$ *for some K3 surface X. These are the cyclic groups of order at most 8 and furthermore*

$$(\mathbb{Z}/2\mathbb{Z})^2,\ (\mathbb{Z}/2\mathbb{Z})^3,\ (\mathbb{Z}/2\mathbb{Z})^4,\ (\mathbb{Z}/3\mathbb{Z})^2,\ (\mathbb{Z}/4\mathbb{Z})^4,\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z},\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}.$$

REFERENCES

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[Nik14] Viacheslav Nikulin, Elliptic fibrations on K3 surfaces, *Proc.Edinb.Math.Soc.(2)*, 57(1), p.253-267, 2014