Proper pushforward of coherent sheaves

Wouter Zomervrucht, March 19, 2014

1. Čech cohomology

Definition 1.1. Let X be a ringed space and $\mathcal{F}$ an $\mathcal{O}_X$-module. Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of X. Put

$$C^r(\mathcal{U}, \mathcal{F}) := \prod_{i_0, \ldots, i_r \in I} \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_r})$$

and define maps

$$C^r(\mathcal{U}, \mathcal{F}) \to C^{r+1}(\mathcal{U}, \mathcal{F}), \quad (a_{i_0 \ldots i_r})_{i_0, \ldots, i_r} \mapsto \Big(\sum_{j=0}^{r+1} (-1)^j a_{i_0 \ldots \hat{i}_j \ldots i_{r+1}} |_{U_{i_0} \cap \ldots \cap U_{i_{r+1}}} \Big)_{i_0, \ldots, i_{r+1}}.$$

The *r-th Čech cohomology group* of $\mathcal{F}$ relative to $\mathcal{U}$, denoted $\check{\mathrm{H}}^r(\mathcal{U}, \mathcal{F})$, is the r-th cohomology group of the cochain complex $C^\bullet(\mathcal{U}, \mathcal{F})$. ♦

The purpose of Čech cohomology is to compute the 'true' cohomology. For simplicity one may endow the index set I with a total ordering $<$ and consider the *ordered complex*

$$C^r_<(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \ldots < i_r \in I} \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_r})$$

with maps $C^r_<(\mathcal{U}, \mathcal{F}) \to C^{r+1}_<(\mathcal{U}, \mathcal{F})$ as before. The cohomology of $C^\bullet_<(\mathcal{U}, \mathcal{F})$ is canonically isomorphic to the usual Čech cohomology.

Theorem 1.2. *Let X be a scheme and $\mathcal{U} = (U_i)_{i \in I}$ an open cover of X such that $U_{i_0} \cap \ldots \cap U_{i_r}$ is affine for all $r \geq 0$ and $i_0, \ldots, i_r \in I$. Then for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and all $r \geq 0$ we have $\check{\mathrm{H}}^r(\mathcal{U}, \mathcal{F}) \cong \mathrm{H}^r(X, \mathcal{F})$ as $\mathcal{O}_X(X)$-modules.*

Proof. Let $\mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$ and let $\mathrm{Tot}^\bullet$ denote the total complex associated to the double complex $C^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$. We have morphisms of complexes

$$C^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow C^\bullet(\mathcal{U}, \mathcal{I}^0) \longrightarrow \mathrm{Tot}^\bullet \longleftarrow C^0(\mathcal{U}, \mathcal{I}^\bullet) \longleftarrow \Gamma(X, \mathcal{I}^\bullet).$$

For an injective sheaf $\mathcal{I}$ and $r > 0$ we have $\check{\mathrm{H}}^r(\mathcal{U}, \mathcal{I}) = 0$, so the row $C^\bullet(\mathcal{U}, \mathcal{I}^q)$ is a resolution of $\Gamma(X, \mathcal{I}^q)$ for all $q \geq 0$. Hence $\Gamma(X, \mathcal{I}^\bullet) \to \mathrm{Tot}^\bullet$ is a quasi-isomorphism. On the other hand, if U is affine and $r > 0$ then $\mathrm{H}^r(U, \mathcal{F}) = 0$. So the column $C^p(\mathcal{U}, \mathcal{I}^\bullet)$ is a resolution of $C^p(\mathcal{U}, \mathcal{F})$ for all $p \geq 0$, and $C^\bullet(\mathcal{U}, \mathcal{F}) \to \mathrm{Tot}^\bullet$ is a quasi-isomorphism as well. ■

An important application is the computation of the cohomology of projective space.

Theorem 1.3. *Let A be a ring, $n \geq 0$ and $d \in \mathbb{Z}$. Then*

$$\mathrm{H}^r(\mathbb{P}^n_A, \mathcal{O}(d)) = \begin{cases} A[x_0, \ldots, x_n]_d & \text{if } r = 0, \\ \big(\frac{1}{x_0 \cdots x_n} A[\frac{1}{x_0}, \ldots, \frac{1}{x_n}]\big)_d & \text{if } r = n, \\ 0 & \text{otherwise.} \end{cases}$$ ♦

See [Stacks `01ED`, `01E0`, `01FG`, `01X8`, `01XS`] for more details.

2. Proper pushforward of coherent sheaves

Theorem 2.1. *Let $f\colon X \to S$ be a proper morphism of locally noetherian schemes. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then for all $r \geq 0$ the higher direct image $\mathrm{R}^r f_* \mathcal{F}$ is a coherent $\mathcal{O}_S$-module.*

Proof. We may assume $S = \operatorname{Spec} A$ for some noetherian ring A.

First case: the structure map $f\colon \mathbb{P}^n_A \to S$. We have $\mathrm{R}^r f_* \mathcal{F} = \mathrm{H}^r(\mathbb{P}^n_A, \mathcal{F})^\sim$, and $\mathrm{H}^r(\mathbb{P}^n_A, \mathcal{F})$ is finitely generated by our computation of the cohomology of projective space.

Second case: $f\colon X \to S$ projective. Write $f = g \circ i$ with $i\colon X \to \mathbb{P}^n_A$ is a closed immersion and $g\colon \mathbb{P}^n_A \to S$ the structure map. Then $i_*\mathcal{F}$ is coherent and $\mathrm{R}^q i_* \mathcal{F} = 0$ for $q > 0$. So the Grothendieck spectral sequence for $f_* = g_* i_*$ gives $\mathrm{R}^p f_* \mathcal{F} = \mathrm{R}^p g_*(i_* \mathcal{F})$, which is coherent.

General case: $f\colon X \to S$ proper. By Chow's lemma there is a scheme Y with projective maps $\pi\colon Y \to X$ and $g\colon Y \to S$ such that $g = f \circ \pi$ and such that there is a dense open $U \subset X$ for which $\pi\colon \pi^{-1}U \to U$ is an isomorphism. We apply noetherian induction.

The kernel and cokernel of $\mathcal{F} \to \pi_* \pi^* \mathcal{F}$ are coherent with support on $X \setminus U$ so have coherent higher direct images; so it suffices to prove that $\pi_* \pi^* \mathcal{F}$ has coherent higher direct images. Observe that $\mathrm{R}^p f_* \mathrm{R}^q \pi_*(\pi^* \mathcal{F})$ is coherent for $p \geq 0, q > 0$, and $\mathrm{R}^{p+q} g_*(\pi^* \mathcal{F})$ is coherent for all $p, q \geq 0$. From the Grothendieck spectral sequence for $g_* = f_* \pi_*$ it follows that $\mathrm{R}^p f_* \mathrm{R}^q \pi_*(\pi^* \mathcal{F})$ is coherent for $q = 0$ as well. ■

3. Serre duality

Definition 3.1. Let X be an n-dimensional scheme proper over a field k. A *dualizing sheaf* for X is a coherent sheaf ω on X endowed with a *trace* map $t\colon \mathrm{H}^n(X, \omega) \to k$ such that

$$\operatorname{Hom}(\mathcal{F}, \omega) \times \mathrm{H}^n(X, \mathcal{F}) \longrightarrow \mathrm{H}^n(X, \omega) \xrightarrow{t} k$$

is a perfect pairing for all $\mathcal{F} \in \operatorname{Coh} X$. ♦

Dualizing shaeves exist and are uniquely unique. We obtain an isomorphism

$$\mathrm{H}^0(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega)) \cong \mathrm{H}^n(X, \mathcal{F})^\vee.$$

for all $\mathcal{F} \in \operatorname{Coh} X$.

Theorem 3.2. *Let X be an n-equidimensional scheme projective over a field k with dualizing sheaf ω.*

- *There are maps*
$$\varphi_i\colon \operatorname{Ext}^i(\mathcal{F}, \omega) \to \mathrm{H}^{n-i}(X, \mathcal{F})^\vee$$
for $i \geq 0$ and $\mathcal{F} \in \operatorname{Coh} X$.
- *If X is Cohen–Macaulay, then all φ_i are isomorphisms.*
- *If X is smooth, then $\omega \cong \Omega^n_{X/k}$.* ♦

Observe that for $\mathcal{F}$ locally free of finite rank, $\operatorname{Ext}^i(\mathcal{F}, \omega) = \mathrm{H}^i(X, \mathcal{H}om_{\mathcal{O}_X}(\mathbb{F}, \omega))$.

Complexes of sheaves offer two advantages here: if a dualizing complex exists, there will automatically be isomorphisms φ_i as in the theorem; and we have duality between H^i and H^{n-i} for all coherent $\mathcal{F}^\bullet$, not just the locally free ones.

Example 3.3. Let k be a field and R a finite-dimensional k-algebra. Then $\operatorname{Spec} R$ is proper over k and 0-dimensional. Its dualizing sheaf is $\operatorname{Hom}_k(R, k)^\sim$. This is locally free of rank 1 if and only if R is Gorenstein. ♦