Proper pushforward of coherent sheaves
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1. Čech cohomology

Definition 1.1. Let $X$ be a ringed space and $\mathcal{F}$ an $\mathcal{O}_X$-module. Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of $X$. Put

$$C^r(\mathcal{U}, \mathcal{F}) := \prod_{I_{i_0, \ldots, i_r} \in I} \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_r})$$

and define maps

$$C^r(\mathcal{U}, \mathcal{F}) \rightarrow C^{r+1}(\mathcal{U}, \mathcal{F}), \quad (a_{i_0, \ldots, i_r})_{i_0, \ldots, i_r} \mapsto \left( \sum_{j=0}^{r+1} (-1)^{j} a_{i_0, \ldots, i_{j-1}, i_{j+1}, \ldots , i_r} \right)_{i_0, \ldots, i_{r+1}}.$$

The $r$-th Čech cohomology group of $\mathcal{F}$ relative to $\mathcal{U}$, denoted $\check{H}^r(\mathcal{U}, \mathcal{F})$, is the $r$-th cohomology group of the cochain complex $C^\bullet(\mathcal{U}, \mathcal{F})$.

The purpose of Čech cohomology is to compute the ‘true’ cohomology. For simplicity one may endow the index set $I$ with a total ordering $<$ and consider the ordered complex

$$C^r_<(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \ldots < i_r} \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_r})$$

with maps $C^r_< (\mathcal{U}, \mathcal{F}) \rightarrow C^r_{<+1}(\mathcal{U}, \mathcal{F})$ as before. The cohomology of $C^r_< (\mathcal{U}, \mathcal{F})$ is canonically isomorphic to the usual Čech cohomology.

Theorem 1.2. Let $X$ be a scheme and $\mathcal{U} = (U_i)_{i \in I}$ an open cover of $X$ such that $U_{i_0} \cap \ldots \cap U_{i_r}$ is affine for all $r \geq 0$ and $i_0, \ldots, i_r \in I$. Then for all quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and all $r \geq 0$ we have $\check{H}^r(\mathcal{U}, \mathcal{F}) = H^r(X, \mathcal{F})$ as $\mathcal{O}_X$-modules.

Proof. Let $\mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$ and let $\text{Tot}^\bullet$ denote the total complex associated to the double complex $C^\bullet(\mathcal{U}, \mathcal{I}^\bullet)$. We have morphisms of complexes

$$C^\bullet(\mathcal{U}, \mathcal{F}) \rightarrow C^\bullet(\mathcal{U}, \mathcal{I}^0) \rightarrow \text{Tot}^\bullet \leftarrow C^0(\mathcal{U}, \mathcal{I}^\bullet) \leftrightarrow \Gamma(X, \mathcal{I}^\bullet).$$

For an injective sheaf $\mathcal{I}$ and $r > 0$ we have $\check{H}^r(\mathcal{U}, \mathcal{I}) = 0$, so the row $C^\bullet(\mathcal{U}, \mathcal{I}^q)$ is a resolution of $\Gamma(X, \mathcal{I}^q)$ for all $q \geq 0$. Hence $\Gamma(X, \mathcal{I}^\bullet) \rightarrow \text{Tot}^\bullet$ is a quasi-isomorphism. On the other hand, if $\mathcal{U}$ is affine and $r > 0$ then $H^r(\mathcal{U}, \mathcal{F}) = 0$. So the column $C^p(\mathcal{U}, \mathcal{I}^\bullet)$ is a resolution of $C^p(\mathcal{U}, \mathcal{F})$ for all $p \geq 0$, and $C^\bullet(\mathcal{U}, \mathcal{F}) \rightarrow \text{Tot}^\bullet$ is a quasi-isomorphism as well. ♦

An important application is the computation of the cohomology of projective space.

Theorem 1.3. Let $A$ be a ring, $n \geq 0$ and $d \in \mathbb{Z}$. Then

$$H^r(\mathbb{P}^n_A, \mathcal{O}(d)) = \begin{cases} A[x_0, \ldots, x_n]^d & \text{if } r = 0, \\ (\frac{1}{x_0} \cdots x_n A[\frac{1}{x_0}, \ldots, \frac{1}{x_n}]^d & \text{if } r = n, \\ 0 & \text{otherwise.} \end{cases}$$

See [Stacks 01ED, 01EO, 01FG, 01XS, 01XS] for more details.
2. Proper pushforward of coherent sheaves

**Theorem 2.1.** Let $f : X \to S$ be a proper morphism of locally noetherian schemes. Let $\mathcal{F}$ be a coherent $O_X$-module. Then for all $r \geq 0$ the higher direct image $R^r f_* \mathcal{F}$ is a coherent $O_S$-module.

**Proof.** We may assume $S = \text{Spec} A$ for some noetherian ring $A$.

First case: the structure map $f : \mathbb{P}^n_A \to S$. We have $R^0 f_* \mathcal{F} = H^0(\mathbb{P}^n_A, \mathcal{F})$, and $H^0(\mathbb{P}^n_A, \mathcal{F})$ is finitely generated by our computation of the cohomology of projective space.

Second case: $f : X \to S$ projective. Write $f = g \circ i$ with $i : X \to \mathbb{P}^n_A$ a closed immersion and $g : \mathbb{P}^n_A \to S$ the structure map. Then $i_* \mathcal{F}$ is coherent and $R^q i_* \mathcal{F} = 0$ for $q > 0$. So the Grothendieck spectral sequence for $f_* = g_* i_*$ gives $R^p f_* \mathcal{F} = R^p g_* (i_* \mathcal{F})$, which is coherent.

General case: $f : X \to S$ proper. By Chow’s lemma there is a scheme $Y$ with projective maps $\pi : Y \to X$ and $g : Y \to S$ such that $g = f \circ \pi$ and such that there is a dense open $U \subset X$ for which $\pi^{-1} U \to U$ is an isomorphism. We apply noetherian induction.

The kernel and cokernel of $\mathcal{F} \to \pi_* \pi^* \mathcal{F}$ are coherent with support on $X \setminus U$ so have coherent higher direct images; so it suffices to prove that $\pi_* \pi^* \mathcal{F}$ has coherent higher direct images. Observe that $R^p g_* R^q i_* \mathcal{F}$ is coherent for $p \geq 0, q > 0$, and $R^{p+q} g_* (\pi^* \mathcal{F})$ is coherent for all $p, q \geq 0$. From the Grothendieck spectral sequence for $g_* = f_* \pi_*$ it follows that $R^p f_* R^q \pi_* (\pi^* \mathcal{F})$ is coherent for $q = 0$ as well.

3. Serre duality

**Definition 3.1.** Let $X$ be an $n$-dimensional scheme proper over a field $k$. A dualizing sheaf for $X$ is a coherent sheaf $\omega$ on $X$ endowed with a trace map $\iota : H^0(X, \omega) \to k$ such that

$$\text{Hom}(\mathcal{F}, \omega) \times H^0(X, \mathcal{F}) \longrightarrow H^0(X, \omega) \xrightarrow{\iota} k$$

is a perfect pairing for all $\mathcal{F} \in \text{Coh } X$.

Dualizing sheaves exist and are uniquely unique. We obtain an isomorphism

$$H^0(X, \text{Hom}_{O_X}(\mathcal{F}, \omega)) \cong H^n(X, \mathcal{F})^\vee.$$

for all $\mathcal{F} \in \text{Coh } X$.

**Theorem 3.2.** Let $X$ be an $n$-equidimensional scheme projective over a field $k$ with dualizing sheaf $\omega$.

- There are maps $\varphi_i : \text{Ext}^i(\mathcal{F}, \omega) \to H^{n-i}(X, \mathcal{F})^\vee$

  for $i \geq 0$ and $\mathcal{F} \in \text{Coh } X$.

- If $X$ is Cohen–Macaulay, then all $\varphi_i$ are isomorphisms.

- If $X$ is smooth, then $\omega \cong \Omega^n_{X/k}$

Observe that for $\mathcal{F}$ locally free of finite rank, $\text{Ext}^i(\mathcal{F}, \omega) = H^i(X, \text{Hom}_{O_X}(\mathcal{F}, \omega))$.

Complexes of sheaves offer two advantages here: if a dualizing complex exists, there will automatically be isomorphisms $\varphi_i$ as in the theorem; and we have duality between $H^i$ and $H^{n-i}$ for all coherent $\mathcal{F}$, not just the locally free ones.

**Example 3.3.** Let $k$ be a field and $R$ a finite-dimensional $k$-algebra. Then $\text{Spec } R$ is proper over $k$ and $0$-dimensional. Its dualizing sheaf is $\text{Hom}_k(R, k)^\vee$. This is locally free of rank $1$ if and only if $R$ is Gorenstein.

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