The Derived Category of D-modules

Derived categories

Motto: Complexes good, homology of complexes bad.

To justify, look a bit at algebraic topology:

Problem: Define invariant of simplicial complexes that decides when \( X \sim Y \) (homotopically equivalent).

Recall: This means \( \exists f: X \to Y \) continuous s.t.

\[
\begin{align*}
gof &= \text{id}_X \\
g \circ f &= \text{id}_Y
\end{align*}
\]

Idea: in homology, we know already that \( X \sim Y \Rightarrow H_i(X) \cong H_i(Y) \).

Bad, because \( \exists X, Y \text{ with } H_i(X) \cong H_i(Y) \) s.t. \( X \nless Y \).

\( \Rightarrow \) homology only gives limited information about homotopy type.

\[X = S^3 \times \mathbb{RP}^3\]

\[Y = \mathbb{RP}^2 \times S^2\]

same universal cover \( S^3 \times S^3 \) \( \pi_1(X) = \pi_1(Y) = \mathbb{Z}_2 \) \( \neq \)

\( X = S^4 \vee (S^2 \times S^2) \) same homology, but different.

\( Y = \mathbb{CP}^2 \lor \mathbb{CP}^2 \) characteristic classes.
But what is homology? Recall it is defined from a simplicial chain complex

\[ C_k(X) \to C_{k-1}(X) \to \cdots \]

\[ H_k(X) = \ker \partial_k / \text{im} \partial_{k+1} \]

Then we must suggest we should take as invariant the role \( C_*(X) \).

**Theorem (Whitehead)** \( X \simeq Y \) iff \( \exists \) a simplicial complex \& simplices \( f \)

maps \( \tilde{f} : \tilde{Z} \to X \) \( \tilde{g} : \tilde{Z} \to Y \) \( \tilde{f} \) \( \tilde{g} \)

\( f_* : C_*(Z) \to C_*(X) \) \( g_* : C_*(Z) \to C_*(Y) \)

Recall: Quasi-isomorphisms \( f_* : H_*(Z) \to H_*(X) \) \( g_* : H_*(Z) \to H_*(Y) \)

Note: 1. We need to keep the info on \( C_\ast \) to get homotopic invariant i.e. not enough to have \( H_*(X) \cong H_*(Y) \) as \( \text{u.p.} \).

2. We need a third space \( \tilde{Z} \)

Why? Homology equivalence might mess up the simplicial structure, so need refinement.

3. Same statement also for cochains \( C^\ast(X) \)

4. Same statement also for cochains \( C^\ast(X) \)

... and cohomology

5. Quasi-isomorphisms are not usually invertible.
Back to algebraic geometry

X top space

\( F \in \text{Ab}(X) \) - sheaves of abelian groups on \( X \)

\( \Gamma(X, \cdot) : \text{Ab}(X) \rightarrow \text{Ab} \) \text{ global section functor}

left-exact

Want invariants on \( \text{Ab}(X) \) - obtained as coh of a complex

\( \text{(???) bbb ???) \)

\( \text{How? \ Replace \ F \ by \ resolution} \)

\[ 0 \rightarrow F \rightarrow I^0 \rightarrow I^1 \rightarrow \ldots \]

Not just any resolution, but an injective resolution

\( I^i = \text{injective objects of } \text{Ab}(X) \)

\( I \text{ injective : } \text{Hom}(\cdot, I) \text{ is exact} \)

(usually just left-exact)

Can we even write this resolution?

Fact \( \text{Ab}(X) \) has enough injectives i.e. \( \forall A \in \text{Ab}(X) \)

\( \exists I \in \text{Ab}(X) \text{ inj s.t. } 0 \rightarrow A \rightarrow I \)

\( \Rightarrow \ A \text{ object in } \text{Ab}(X) \text{ has an injective resolution.} \)
Coherent sheaf

\[ H^i(X, \mathcal{F}) = R^i \mathcal{H}om(X, \mathcal{F}) = h^i(\mathcal{H}om(X, \mathcal{F})) \]

So far, \( X \) just top sp; \( \mathcal{F} \) sheaf of groups.

Extra structure \( (X, \mathcal{O}_X) \) ringed space

\( \mathcal{F} \) \( \mathcal{O}_X \)-module

\[ \mathcal{H}om(X, \mathcal{F}) : \text{Mod}_{\mathcal{O}_X}(X) \to \text{Ab} \]

Fact: \( \text{Mod}_{\mathcal{O}_X}(X) \) has enough injectives

So again take inj resolutions etc.

One can check that calculating

\[ R^i \mathcal{H}om(X, \mathcal{F}) \] with inj resol in \( \text{Mod}_{\mathcal{O}_X}(X) \)

yields the same cohomology functors

\[ H^i(X, \mathcal{F}) \].

Motto: perhaps better to keep the complexes instead of the coh groups.
Main object of study are complexes = want a category.

**The homotopy category.**

Base abelian category $A$.

Construct new category with

- objects = complexes of objects of $A$.
- maps = chain maps.

$A^i = \cdots \to A^{i-1} \xrightarrow{d_i} A^i \xrightarrow{d_i} A^{i+1} \to \cdots$

$f: A^i \to B^i = \text{collection } \{ f^i: A^i \to B^i \}_{i \in \mathbb{Z}}$ such that

the obvious square commutes:

$$
\begin{array}{ccc}
A^i & \xrightarrow{d_i} & A^{i+1} \\
\downarrow{f^i} & & \downarrow{f^{i+1}} \\
B^i & \xrightarrow{d_i} & B^{i+1}
\end{array}
$$

Want to do something similar to homotopy theory.

Back to our example: $X, Y$ with simplicial complexes $\Delta$, homotopic maps $f, g: Y \to X$.

Get induced maps $f^*: C^i(Y) \to C^i(X)$ and $h^*: C^i(X) \to C^{i-1}(Y)$ s.t.

$$
[f^* - g^*] = d^i \circ h^* + h^{i+1} \circ d^i
$$

Want smooth like: homotopic maps are equal.
In a arbitrary ab cat, set
def 
\[ f, g : A \to B \] homotopic \[ f \sim g \] as above.

Fact:
\[ f \sim g \] is an equivalence relation \[ \sim \]
\[ f_1 \sim g_1 \]
\[ f_2 \sim g_2 \]
\[ f_1 \circ g_1 = f_1 \circ f_2 \circ g_1 \circ g_2 . \]

Def
A abelian category
\[ K(A) \] homotopy category
- objects: \[ A^* \], \[ A \in \text{obj}(A) \]
- morphisms: chain maps \[ \sim \]

Properties:
1. Another great reason to treat homotopy maps as equal:
   injective resolutions
   \[ A, B \in \text{obj}(A) \]
   \[ f : A \to B \] morphisms
   \[ o \to A \to I_A^* \]
   \( f \) can be lifted to \( \tilde{f} : I_A^* \to I_B^* \)
   \[ o \to B \to I_B^* \]
   \( \tilde{f} \) not unique, but \( \tilde{f}_1 \sim \tilde{f}_2 \)
   \[ \Rightarrow \text{Hom}_A(A, B) \cong \text{Hom}_{K(A)}(I_A^*, I_B^*) \]

2. \( K(A) \) not abelian, have some other notions for exactness
   (triangulated structure)
Not quite done. Want also to express a Whitehead-Thomason-type result in a novel way.

Recall from the simplicial complexes example

\[
\begin{array}{c}
\text{X} \to \text{Y} \\
\text{ quasi-iso } \downarrow \text{ quasi-iso }
\end{array}
\]

\[ C'(Z) \cong C'(Y) \]

Want to be able to invert these.
So pretend that \( g \cdot i \) are iso!
This is localization.

\[ \text{Def } A \text{ abelian category} \]
\[ D(A) \text{ derived category} \]
\[ = \text{pretend } g \cdot i \text{ in } K(A) \text{ are isomorphisms} \]

Morphisms in \( D(A) \): \( A' \to B' \) in the roof

\[
\begin{array}{c}
C' \\
\downarrow \\
A' \\
\end{array}
\]

\[ f^* \text{ Mor } K(A) \]

\[ f \cdot g^* \text{ Mor } K(A) \]

\[ f \cdot k_i \]

\[ \text{roof represents } g \cdot f \cdot i \text{ even though } f^{-1} \text{ doesn't exist} \]
\[ \text{just like } \frac{3}{1} \text{ represents } 3 \cdot 1^{-1} \in \mathbb{Q} \]
\[ \text{even though } 4^{-1} \notin \mathbb{Z} \]
No worries, the construction is sound!

\[ \text{Gabriel-Zisman Thm: } \mathcal{D}(\mathcal{A}) \text{ exists.} \]

We have functor \( \mathcal{Q}: \mathcal{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{A}) \)

It is universal!

\[ \mathcal{Q}(A^\prime) = \text{id} \]

\[ \mathcal{Q}(f: A^\prime \to B^\prime) = \text{id} \]

Why bother? Now we have a way of representing objects in \( \mathcal{A} \)
as inj resolutions and we have a nice way of dealing with their morphisms too.

Advantage: The right derived functors can be expressed in a unified way.
More precisely: For simplicity, work only w/ bounded complexes i.e. \( A \) n.t. \( \text{H}^i(A) = 0 \) for \( i > 0 \).

\[ \text{Sp ox } A \text{ has enough injectives.} \]

Denote by \( K^b(A) \) the bounded categories.

\[ D^b(A) \]

\[ \text{Sp ox } A \text{ has enough injectives.} \]

Then every bounded below \( A \in K^b(A) \) is \( \alpha \)-trivial quasi-isomorphic to complex of injectives.

Denote by \( K^b(\text{inj}(A)) \) complexes of injectives.

Then we have an equivalence

\[ K^b(\text{inj}(A)) \xrightarrow{\alpha} D^b(A) \]

Left-exact functor \( F: A \rightarrow B \) \( \forall A, B \)

\[ \Rightarrow \text{ have exact functor } F: K^b(\text{inj}(A)) \rightarrow K^b(B) \]

If \( A \) has enough injectives, \( F \) is left-exact.

\[ RF : D^b(A) \xrightarrow{Q^{-1}} K^b(\text{inj}(A)) \xrightarrow{F} K^b(B) \xrightarrow{Q} D^b(B) \]

What is \( RF(A) \), \( A \in A \).

\[ I^* = Q^{-1}(A) \]

Apply \( F \) to \( I^* \), get \( RF(A) \).
Take cohomology: \[ H^i(\mathcal{F}(A)) = H^i(F(I^*)) = R^iF(A) \]

This can be done for complexes & \[ RF(A^*) \] is a complex of a bunch of objects.

Note: for left exact need injectives
  for right exact need projectives.

lucky, for our D-modules, we have both!
$\mathcal{D}_x$-quasi-coherent $\mathcal{D}_x$-modules

Denote by $\mu(\mathcal{D}_x)$ - category of $\mathcal{D}_x$-quasi-coherent $\mathcal{D}_x$-modules.

$\mu(\mathcal{D}_x)$ has enough injectives:

Any $M \in \mu(\mathcal{D}_x)$ can be embedded in an inj $\mathcal{D}_x$-quasi-coherent module.

Proof (Sketch)

$\{X_i\}$ finite open affine cover of $X$

$\phi_i : X_i \rightarrow X$

$M_i := \phi_i^* M$ $M_i = M(X_i)$

Can find injective $D(X_i)$-module $J_i$ containing $M_i$ as $D(X_i)$-submodule

by cat of $R$-modules has enough inj ($R$ not necessarily comm.)

Take $\psi_j$ = associated $\mathcal{D}_{X_j}$-module i.e.

a sheaf on $X_j$ s.t.

$\psi_j(U_j)$ is a $D(U_j)$-module

$\psi_j|U_j \subset X_j$ open

$X_i$ affine no dc

$\psi_j$ injective
Then we have \[ 0 \to M_i \to Y_i \]

\[ j_{i*} \text{ preserves } \& \text{ c} \]

\[ \& \text{ is exact} \]

\[ (X \text{ affine}) \]

\[ 0 \to j_{i*}(M_i) \to j_{i*}(Y_i) \]

These direct images are still $D_x$-modules:

\[ U \in X \text{ open} \]

\[ j_{i*}(M_i)(U) = M_i(j_i^*(U)) = M_i(U \cap X_i) \]

\[ j_{i*}(Y_i)(U) = Y_i(j_i^*(U)) = Y_i(U \cap X_i) \]

Now $D_x(U \cap X_i)$ acts on $M_i(U \cap X_i)$ via $D_{X_i}(U \cap X_i)$.

\[ j_{i*}(Y_i) \text{ injective} \]

\[ \text{(We } j_{i*} \text{ left-adj. to } j_i^* \text{)} \]

\[ j_{i*} \text{ exact} \]

\[ Y_i \text{ inj} ]

So \[ U \xrightarrow{\ni \circ j_i} j_{i*} M_i = j_{i*} M_i \]

\[ Y = \oplus j_{i*} j_{i*} \text{ injective} \]

\[ \rho \circ \alpha : M \to j_{i*} M_i \to j_{i*} Y_i \]

\[ \text{hom inj} \]

\[ M \to Y \]

\[ \text{Module } \]
\[ \mu(D_X) \text{ has enough projectives} \]

\[ X \text{ quasi-proj, } F \in \mu(D_X) \]

Then \( F \) is a quotient of locally free \( D_X \)-module.

So we have \( i : X \to P \) locally closed proj embedding

ETS: \( i^*_P F = \text{quotient of locally free } D_P \text{-module} \]

\[ b/c \text{ then } Q/\pi_i X \text{ is locally free over } O_X \]

\& \( F \) quotient of \( D_X \otimes \Omega_X(Q/\pi_i X) \)

\[ Q \to i^*_P F \to 0 \]

\[ i^* \text{ exact } \& \]

\[ D_X \otimes \Omega_X(Q/\pi_i X) \to F \to 0 \]

Now \( i^*_P \) is an \( O_p \) = \( i^*_P \) inductive limit of \( O_p \)-coherent submodules \( F_i \).

Then \( H_i \text{ for } \cdot \otimes O_p(m_i) \to F_i \to 0 \)

\[ \Rightarrow Q := \bigoplus \bigoplus \bigoplus_{O_p(m_i)} \to F_i \to 0 \]

Is \( Q \) locally free?

Take \{ \Phi_i \} open cover of \( P \) (with hom coord in \( \neq 0 \))

Then \( O_p(m_i)|_{\Phi_i} \text{ is free } \Rightarrow Q|_{\Phi_i} \text{ free } \Rightarrow Q \text{ locally free} \)