The Gieseker conjecture

From this lecture onwards, we will study D-modules in positive characteristic. But for the sake of motivation, we first recall some facts from characteristic zero. Let $X/\mathbb{C}$ be a proper smooth variety. We have seen equivalences

\[ \text{stratified bundles on } X \overset{(1)}{\rightarrow} \text{stratified bundles on } X^{\text{an}} \overset{(2)}{\rightarrow} \text{representations of } \pi_1(X^{\text{an}}) \]

where (1) comes from the GAGA theorems and (2) is the Riemann–Hilbert correspondence. We had trouble in making this fully algebraic since, algebraically, only the étale fundamental group $\hat{\pi}_1^{\text{et}}(X) = \hat{\pi}_1(X^{\text{an}})$ is available. Nevertheless we can say something, due to the following forgotten-and-rediscovered fact.

**Theorem 1.1 (Malčev–Grothendieck).** Let $G \rightarrow H$ be a homomorphism of finitely generated groups. If $\hat{G} \rightarrow \hat{H}$ is an isomorphism, then the pullback functor $\text{Rep}_C H \rightarrow \text{Rep}_C G$ is an equivalence.

**Theorem 1.2.** Let $X/\mathbb{C}$ be a proper smooth variety. Then

- $\pi_1^{\text{et}}(X)$ is abelian if and only if every irreducible stratified bundle on $X$ is one-dimensional, and
- $\pi_1^{\text{et}}(X) = 1$ if and only if every stratified bundle on $X$ is trivial.

**Proof.** Suppose $\pi_1^{\text{et}}(X)$ is abelian. The groups $\pi = \pi_1(X^{\text{an}})$ and $\pi^{\text{ab}} = \pi/\langle \pi, \pi \rangle$ are finitely generated and $\pi \rightarrow (\pi^{\text{ab}})^\ast$ is an isomorphism by assumption, so all representations of $\pi$ come from $\pi^{\text{ab}}$. Due to Schur’s lemma, all irreducible representations of an abelian group are one-dimensional.

If all irreducible representations of $\pi_1(X^{\text{an}})$ are one-dimensional, all representations have an eigenvector, hence all representations are conjugate to one on upper-triangular matrices. Let $G$ be a finite quotient of $\pi_1(X^{\text{an}})$ and $\rho$ the regular representation of $G$, which we may assume upper-triangular. For all $g, h \in G$ the matrix $\rho([g, h])$ is unipotent, i.e. has 1s on the diagonal. As $G$ is finite, $\rho([g, h])$ has finite order, implying $\rho([g, h]) = 1$. The regular representation is faithful, so $[g, h] = 1$ and $G$ is abelian. Then also $\pi_1^{\text{et}}(X) = \hat{\pi}_1(X^{\text{an}})$ is abelian.

Suppose $\pi_1^{\text{et}}(X) = 1$. Then $\pi_1(X^{\text{an}}) \rightarrow 1$ becomes an isomorphism after passing to profinite completions, so all representations of $\pi_1(X^{\text{an}})$ are trivial. Conversely, if all representations of $\pi_1(X^{\text{an}})$ are trivial, then in particular the regular representations of all finite quotients of $\pi_1(X^{\text{an}})$ are trivial. Regular representations are faithful, hence all those finite quotients are trivial and $\pi_1^{\text{et}}(X) = \hat{\pi}_1(X^{\text{an}}) = 1$.

This leads to the following conjecture in positive characteristic.

**Conjecture 1.3 (Gieseker).** Let $k$ be an algebraically closed field of characteristic $p > 0$ and $X/k$ a proper smooth variety. Then $\pi_1^{\text{et}}(X) = 1$ if and only if every stratified bundle on $X$ is trivial.

The implication $\Leftarrow$ is easy and will be proved in the next lecture. The converse was proved only recently by Esnault–Mehta.
2. F-divided bundles

Let $k$ be an algebraically closed field of characteristic $p > 0$ and $X/k$ a smooth variety. In characteristic zero stratified bundles have a simpler description as integrable connections. In positive characteristic there is a different description.

**Definition 2.1.** The absolute Frobenius of $X$ is the map $F: X \to X$ that is the identity on topological spaces and the $p$-power map on sheaves. It fits into a commutative square

$$
\begin{array}{ccc}
X & \xrightarrow{F} & X \\
\downarrow & & \downarrow \\
\text{Spec } k & \xrightarrow{F} & \text{Spec } k.
\end{array}
$$

If $\mathcal{E}$ is an $\mathcal{O}_X$-module, its Frobenius twist is the $\mathcal{O}_X$-module $F^* \mathcal{E}$.

Observe that the natural map $\lambda: \mathcal{E} \to F^* \mathcal{E}, s \mapsto 1 \otimes s$ is $p$-linear, i.e. for $f \in \mathcal{O}_X$ and $s \in \mathcal{E}$ one has $\lambda(f s) = f^p \lambda(s)$.

**Definition 2.2.** An F-divided bundle on $X$ is a collection of coherent $\mathcal{O}_X$-modules $\mathcal{E}_n$, $n \in \mathbb{N}$, together with $\mathcal{O}_X$-linear isomorphisms $\sigma_n: F^* \mathcal{E}_{n+1} \to \mathcal{E}_n$. A morphism of F-divided bundles $\left(\mathcal{E}_n, \sigma_n\right)_{n \in \mathbb{N}} \to \left(\mathcal{F}_n, \tau_n\right)_{n \in \mathbb{N}}$ is a collection of $\mathcal{O}_X$-linear maps $\alpha_n: \mathcal{E}_n \to \mathcal{F}_n$ such that all squares

$$
\begin{array}{ccc}
F^* \mathcal{E}_{n+1} & \xrightarrow{\sigma_n} & \mathcal{E}_n \\
\downarrow & & \downarrow \\
F^* \mathcal{F}_{n+1} & \xrightarrow{\alpha_n} & \mathcal{F}_n
\end{array}
$$

commute.

In fact the sheaves $\mathcal{E}_n$ are automatically locally free. We omit the proof.

**Theorem 2.3 (Katz).** There is an equivalence of tensor categories

$$
\left\{\text{stratified bundles on } X\right\} \longleftrightarrow \left\{\text{F-divided bundles on } X\right\}.
$$

**Proof.** We just sketch the constructions and omit the verification that they are functorial and quasi-inverse to each other. In one direction, let $\mathcal{E}$ be a stratified bundle. We define subsheaves

$$
\mathcal{E}_n = \{ s \in \mathcal{E} : \theta s = 0 \text{ for all } \theta \in \mathcal{D}^{<p^n} \text{ with } \theta(1) = 0 \},
$$

yielding $\ldots \subseteq \mathcal{E}_1 \subseteq \mathcal{E}_0 = \mathcal{E}$. For $f \in \mathcal{O}_X$ and $s \in \mathcal{E}_n$ one has $f^p s \in \mathcal{E}_n$. Indeed, for $\theta \in \mathcal{D}^{<p^n}$ with $\theta(1) = 0$ we compute $\theta \cdot f^p s = f^p \theta s + \theta(f^p) s = f^p s + 0 \cdot s = 0$. Therefore we can make $\mathcal{E}_n$ into a coherent $\mathcal{O}_X$-module by defining $f \ast s = f^p s$. Moreover, the maps

$$
\sigma_n: F^* \mathcal{E}_{n+1} = \mathcal{O}_X \otimes_{\mathcal{O}_X} F^{-1} \mathcal{E}_{n+1} \to \mathcal{E}_n, \ f \otimes s \mapsto f^p s
$$

are well-defined and $\mathcal{O}_X$-linear. Let’s prove they are isomorphisms. We define a connection $\nabla_n$ on $\mathcal{E}_n$ as follows: if $\theta$ is a derivation, choose a differential operator $\theta' \in \mathcal{D}^{<p^n}$ satisfying
\[ \theta'(f^n) = \theta(f)^n \]
for all \( f \in \mathcal{O}_X \) and set \( \nabla_{n, \theta}(s) = \theta's \). This is independent of the choice of \( \theta' \) and preserves \( \mathcal{E}_n \). Observe that at least locally such \( \theta' \) really exists: in coordinates,

\[ \theta = f_1 \partial_{x_1} + \ldots + f_d \partial_{x_d} \quad \mapsto \quad \theta' = f_1^{(p^n)} \frac{\partial_{x_1}}{p^{n \alpha_1}} + \ldots + f_d^{(p^n)} \frac{\partial_{x_d}}{p^{n \alpha_d}}. \]

Now \( \nabla_n \) is an integrable connection on \( \mathcal{E}_n \) whose \( p \)-curvature is zero, i.e. \( \nabla_{n, \theta} = (\nabla_{n, \theta})^p \).

Moreover, \( \mathcal{E}_n^{\alpha} = \mathcal{E}_{n+1} \). We are done by the following result.

**Lemma 2.4 (Cartier).** Let \( (\mathcal{F}, \nabla) \) be an integrable connection with \( p \)-curvature zero. Then the natural map \( F^* \mathcal{F}^\nabla \to \mathcal{F} \) is an isomorphism.

The proof is a straightforward but nasty computation in local coordinates, which we omit.

Conversely, let \( (\mathcal{E}_n, \sigma_n)_{n \in \mathbb{N}} \) be an \( F \)-divided bundle. We make \( \mathcal{E}_0 \) into a stratified bundle. If \( \theta \) is a differential operator of order less than \( p^n \), locally choose a basis \( u_1, \ldots, u_r \) of \( \mathcal{E}_n \). Via the \( p \)-linear inclusions \( \mathcal{E}_{m+1} \to F^* \mathcal{E}_{m+1} \to \mathcal{E}_m \) this maps to a basis \( \tilde{u}_1, \ldots, \tilde{u}_r \) of \( \mathcal{E}_0 \). Define

\[ \theta(f_1 \tilde{u}_1 + \ldots + f_r \tilde{u}_r) = \theta(f_1) \tilde{u}_1 + \ldots + \theta(f_r) \tilde{u}_r. \]

We claim that this is a well-defined stratification. Let’s just verify that it does not depend on the choice of basis of \( \mathcal{E}_n \). Take a second basis \( v_1, \ldots, v_r \) with base change matrix \( A = (a_{ij}) \). The corresponding base change matrix between \( \tilde{u}_1, \ldots, \tilde{u}_r \) and \( \tilde{v}_1, \ldots, \tilde{v}_r \) is \( A' = (a_{ij}^{p^n}) \). Hence the claim follows by the identity \( \theta(f a^{p^n}) = \theta(f) a^{p^n} \) for \( a, f \in \mathcal{O}_X \).

**Proposition 2.5.** Suppose \( X/k \) is proper. Two \( F \)-divided bundles \( (\mathcal{E}_n, \sigma_n)_{n \in \mathbb{N}} \) and \( (\mathcal{F}_n, \tau_n)_{n \in \mathbb{N}} \) are isomorphic if and only if \( \mathcal{E}_n \) is isomorphic to \( \mathcal{F}_n \) for all \( n \in \mathbb{N} \).

**Proof.** Suppose \( \mathcal{E}_n \cong \mathcal{F}_n \) for all \( n \in \mathbb{N} \). The maps \( \text{End}(\mathcal{E}_{n+1}) \to \text{End}(F^* \mathcal{E}_{n+1}) \to \text{End}(\mathcal{E}_n) \) are \( p \)-linear and injective. Since \( X \) is proper, \( \text{End}(\mathcal{E}_0) \) is a finite-dimensional \( k \)-vector space. Thus there exists \( m \in \mathbb{N} \) such that for all \( n \geq m \) the inclusions \( \text{End}(\mathcal{E}_{n+1}) \to \text{End}(\mathcal{E}_n) \) are bijections. Fix an isomorphism \( \alpha: \mathcal{E}_m \to \mathcal{F}_m \) satisfying \( \alpha(\mathcal{E}_{m+1}) = \mathcal{F}_{m+1} \). (For instance take \( \alpha = \beta^p \), where \( \beta \) is an isomorphism \( \mathcal{E}_{n+1} \to \mathcal{F}_{m+1} \).) Because every automorphism of \( \mathcal{E}_m \) preserves \( \mathcal{E}_{m+1} \), we conclude that in fact every isomorphism \( \mathcal{E}_m \to F^* \mathcal{F}_m \) sends \( \mathcal{E}_{m+1} \) to \( \mathcal{F}_{m+1} \). By induction we see that \( \alpha(\mathcal{E}_n) = \mathcal{F}_n \) for all \( n \geq m \). So \( \alpha \) induces an isomorphism of \( F \)-divided bundles.

**Corollary 2.6 (Katz).** Suppose \( X/k \) is projective. Then a line bundle \( \mathcal{L} \) admits a stratification if and only if it has finite and \( p \)-prime order in \( \text{NS}(X) \). The group of stratifications on \( \mathcal{O}_X \) is isomorphic to the Tate group \( \text{T}_pX = \text{T}_p \text{Pic}(X) = \lim_n \text{Pic}(X)[p^n] \).

**Proof.** The Frobenius twist of a line bundle is \( F^* \mathcal{L} = \mathcal{L}^{\otimes p} \). So the first statement follows from the facts that \( \text{NS}(X) = \text{Pic}(X)/\text{Pic}^o(X) \) is finitely generated and that \( \text{Pic}^o(X) \) is \( p \)-divisible.

As for the second statement, by the preceding proposition the group of stratifications on \( \mathcal{O}_X \) is precisely the group of systems \( (\mathcal{L}_n)_{n \in \mathbb{N}} \) with \( \mathcal{L}_0 \cong \mathcal{O}_X \) and \( F^* \mathcal{L}_{n+1} = \mathcal{L}_{n+1}^{\otimes p} \cong \mathcal{L}_n \).

3